The 13$^{\text{th}}$ Romanian Master of Mathematics Competition

Day 1: Tuesday, October 12, 2021, Bucharest

Language: English

Problem 1. Let T_1, T_2, T_3, T_4 be pairwise distinct collinear points such that T_2 lies between T_1 and T_3, and T_3 lies between T_2 and T_4. Let ω_1 be a circle through T_1 and T_4; let ω_2 be the circle through T_2 and internally tangent to ω_1 at T_1; let ω_3 be the circle through T_3 and externally tangent to ω_2 at T_2; and let ω_4 be the circle through T_4 and externally tangent to ω_3 at T_3. A line crosses ω_1 at P and W, ω_2 at Q and R, ω_3 at S and T, and ω_4 at U and V, the order of these points along the line being P, Q, R, S, T, U, V, W. Prove that $PQ + TU = RS + VW$.

Problem 2. Xenia and Sergey play the following game. Xenia thinks of a positive integer N not exceeding 5000. Then she fixes 20 distinct positive integers a_1, a_2, $\ldots$, a_{20} such that, for each $k = 1, 2, \ldots, 20$, the numbers N and a_k are congruent modulo k. By a move, Sergey tells Xenia a set S of positive integers not exceeding 20, and she tells him back the set $\{a_k \colon k \in S\}$ without spelling out which number corresponds to which index. How many moves does Sergey need to determine for sure the number Xenia thought of?

Problem 3. A number of 17 workers stand in a row. Every contiguous group of at least 2 workers is a *brigade*. The chief wants to assign each brigade a leader (which is a member of the brigade) so that each worker's number of assignments is divisible by 4. Prove that the number of such ways to assign the leaders is divisible by 17.

Each of the three problems is worth 7 marks.
Time allowed $4\frac{1}{2}$ hours.

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Day 2: Wednesday, October 13, 2021, Bucharest

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Problem 4. Consider an integer $n \geq 2$ and write the numbers $1, 2, \ldots, n$ down on a board. A move consists in erasing any two numbers a and b, then writing down the numbers $a + b$ and $|a - b|$ on the board, and then removing repetitions (e.g., if the board contained the numbers $2, 5, 7, 8$, then one could choose the numbers $a = 5$ and $b = 7$, obtaining the board with numbers $2, 8, 12$). For all integers $n \geq 2$, determine whether it is possible to be left with exactly two numbers on the board after a finite number of moves.

Problem 5. Let n be a positive integer. The kingdom of Zoomtopia is a convex polygon with integer sides, perimeter $6n$, and 60° rotational symmetry (that is, there is a point O such that a 60° rotation about O maps the polygon to itself). In light of the pandemic, the government of Zoomtopia would like to relocate its $3n^2 + 3n + 1$ citizens at $3n^2 + 3n + 1$ points in the kingdom so that every two citizens have a distance of at least 1 for proper social distancing. Prove that this is possible. (The kingdom is assumed to contain its boundary.)

Problem 6. Initially, a non-constant polynomial $S(x)$ with real coefficients is written down on a board. Whenever the board contains a polynomial $P(x)$, not necessarily alone, one can write down on the board any polynomial of the form $P(C+x)$ or $C+P(x)$, where C is a real constant. Moreover, if the board contains two (not necessarily distinct) polynomials $P(x)$ and $Q(x)$, one can write $P(Q(x))$ and $P(x)+Q(x)$ down on the board. No polynomial is ever erased from the board.

Given two sets of real numbers, $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$, a polynomial $f(x)$ with real coefficients is (A, B)-*nice* if $f(A) = B$, where $f(A) = \{f(a_i) \colon i = 1, 2, \ldots, n\}$.

Determine all polynomials $S(x)$ that can initially be written down on the board such that, for any two finite sets A and B of real numbers, with $|A| = |B|$, one can produce an (A, B)-nice polynomial in a finite number of steps.

Each of the three problems is worth 7 marks. Time allowed $4\frac{1}{2}$ hours.