Sunday, April 11, 2021

Problem 1. The number 2021 is *fantabulous*. For any positive integer m, if any element of the set $\{m, 2m+1, 3m\}$ is fantabulous, then all the elements are fantabulous. Does it follow that the number 2021^{2021} is fantabulous?

Problem 2. Find all functions $f\colon \mathbb{Q} \to \mathbb{Q}$ such that the equation

$$f(xf(x) + y) = f(y) + x^2$$

holds for all rational numbers x and y.

Here, $\mathbb{Q}$ denotes the set of rational numbers.

Problem 3. Let ABC be a triangle with an obtuse angle at A. Let E and F be the intersections of the external bisector of angle A with the altitudes of ABC through B and C respectively. Let M and N be the points on the segments EC and FB respectively such that $\angle EMA = \angle BCA$ and $\angle ANF = \angle ABC$. Prove that the points E, F, N, M lie on a circle.

Time: 4 hours and 30 minutes
Each problem is worth 7 points

To make this a fair and enjoyable contest for everyone, please do not mention or refer to the problems on the internet or on social media until Tuesday 13 April, 12:00 UTC (05:00 Pacific Daylight Time, 13:00 British Summer Time, 22:00 Australian Eastern Standard Time).

Monday, April 12, 2021

Problem 4. Let ABC be a triangle with incentre I and let D be an arbitrary point on the side BC. Let the line through D perpendicular to BI intersect CI at E. Let the line through D perpendicular to CI intersect BI at F. Prove that the reflection of A in the line EF lies on the line BC.

Problem 5. A plane has a special point O called the origin. Let P be a set of 2021 points in the plane such that

(i) no three points in P lie on a line and

(ii) no two points in P lie on a line through the origin.

A triangle with vertices in P is *fat* if O is strictly inside the triangle. Find the maximum number of fat triangles.

Problem 6. Does there exist a nonnegative integer a for which the equation

$$\left\lfloor \frac{m}{1} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + \cdots + \left\lfloor \frac{m}{m} \right\rfloor = n^2 + a$$

has more than one million different solutions (m, n) where m and n are positive integers?

The expression $\lfloor x \rfloor$ *denotes the integer part (or floor) of the real number* x. *Thus* $\lfloor \sqrt{2} \rfloor = 1$, $\lfloor \pi \rfloor = \lfloor 22/7 \rfloor = 3$, $\lfloor 42 \rfloor = 42$ *and* $\lfloor 0 \rfloor = 0$.

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