The 12th Romanian Master of Mathematics Competition

Day 1 — Solutions

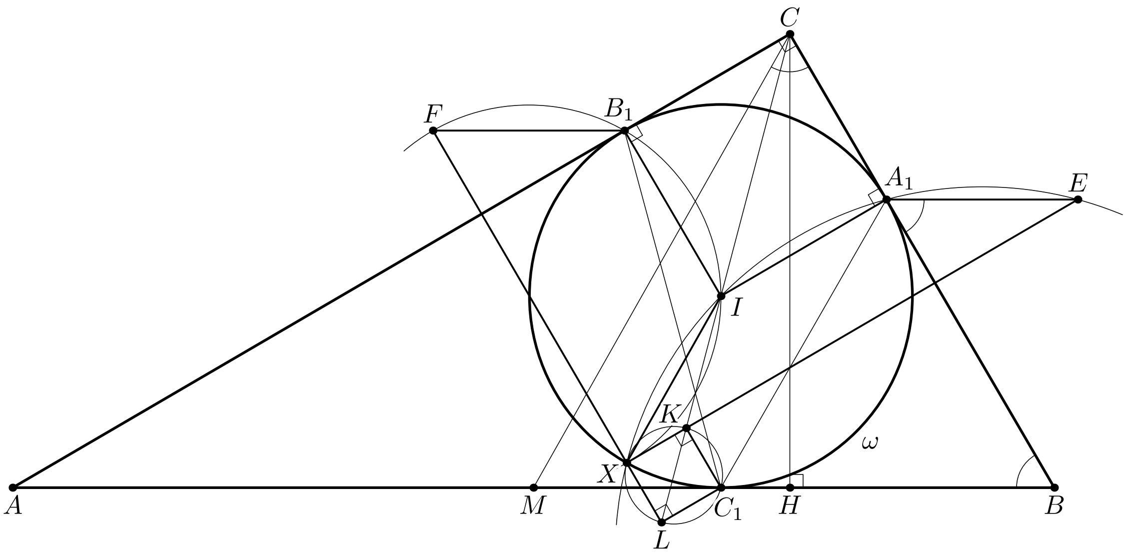
Problem 1. Let ABC be a triangle with a right angle at C, let I be its incentre, and let H be the orthogonal projection of C on AB. The incircle ω of the triangle ABC is tangent to the sides BC, CA, and AB at A_1, B_1, and C_1, respectively. Let E and F be the reflections of C in the lines C_1A_1 and C_1B_1, respectively, and let K and L be the reflections of H in the same lines. Prove that the circles A_1EI, B_1FI, and C_1KL have a common point.

Russia, Dmitry Prokopenko

Solution. The line C_1A_1 is parallel to the external angle bisector of $\angle B$, so the reflection in C_1A_1 maps the segment A_1C to the segment A_1E parallel to AB. Similarly, $B_1F \parallel AB$. Notice also that $A_1E = A_1C = B_1C = B_1F = r$, where r is the inradius of $\triangle ABC$.

Let M be the midpoint of AB. Let X be the point of ω such that $\overrightarrow{IX} \upuparrows \overrightarrow{CM}$. Notice that $\angle EA_1I = 90^\circ + \angle EA_1B = 90^\circ + \angle CBM = 90^\circ + \angle BCM = \angle A_1IX$, and $A_1E = IA_1 = IX$; thus, XIA_1E is an isosceles trapezoid. Hence X lies on the circle IA_1E, and $EX \parallel A_1I$. Similarly, X lies on the circle IB_1F, and $FX \parallel B_1I$. It remains to show that X lies on the circle C_1KL.

Under the symmetry in C_1A_1, the line CH (perpendicular to AB) maps to the line through E perpendicular to BC — i.e., CH maps to EX. Therefore, the projection H of C_1 onto CH maps to the projection K of C_1 onto EX. Similarly, L is the projection of C_1 onto FX. So the quadrilateral C_1KXL is cyclic, due to right angles at K and L.



Comments. **(1)** In fact, the quadrilateral C_1KXL is a square, since $C_1K = C_1H = C_1L$ and $\angle KC_1L = 2\angle A_1C_1B_1 = 90^\circ$.

(2) One can easily see that the points C_1 and X are symmetric in the angle bisector CI. This yields that K and L both lie on CI. One can show that this conclusion in fact holds in any, not necessarily right-angled, triangle.

Problem 2. Let N be a positive integer, and let $\mathbf{a} = (a(1), \ldots, a(N))$ and $\mathbf{b} = (b(1), \ldots, b(N))$ be sequences of non-negative integers, each written on a circle (so we assume $a(i \pm N) = a(i)$ and $b(i \pm N) = b(i)$). We say $\mathbf{a}$ is $\mathbf{b}$-*harmonic*, if each $a(i)$ is the arithmetic mean of the counterclockwise nearest $b(i)$ numbers, the clockwise nearest $b(i)$ numbers, and $a(i)$ itself; that is,

$$a(i) = \frac{1}{2b(i)+1} \sum_{s=-b(i)}^{b(i)} a(i+s). \qquad (*)$$

(A term of $\mathbf{a}$ may appear more than once in the above sum.) Suppose that neither $\mathbf{a}$ nor $\mathbf{b}$ is constant, and that both $\mathbf{a}$ is $\mathbf{b}$-harmonic, and $\mathbf{b}$ is $\mathbf{a}$-harmonic. Prove that more than half of the $2N$ terms across both sequences vanish.

United Kingdom, Dominic Yeo

Solution 1. Let $a = \min_i a(i)$ and let $b = \min_i b(i)$. Since $\mathbf{a}$ is not constant, there exists an i such that $a = a(i) < a(i+1)$.

Claim 1. If $a = a(i) < a(i+1)$, then $b(i) = 0$. Similarly, if $a = a(i) < a(i-1)$, then $b(i) = 0$.

Proof. Otherwise the sum in $(*)$ contains a term $a(i+1) > a$ but no terms smaller than a, so the average is greater than a. $\square$

Claim 1 implies $b = 0$; similarly, $a = 0$. With reference again to Claim 1, $a(i) = b(i) = 0$ for some index i.

Say that $[i, j]$ is an $\mathbf{a}$-segment if $a(i) = a(i+1) = \cdots = a(j) = 0$ but $a(i-1) \neq 0 \neq a(j+1)$; define a $\mathbf{b}$-segment similarly. By Claim 1, the endpoints of any such segment satisfy $a(i) = b(i) = a(j) = b(j) = 0$. Since the sequences are non-constant, each i where $a(i) = 0$ is contained in an $\mathbf{a}$-segment.

Claim 2. Let $[i, j]$ be a $\mathbf{b}$-segment, and let $k \in [i, j]$. Then $a(k) \leq k - i$ (and, similarly, $a(k) \leq j - k$).

Proof. Indeed, since $b(k) = 0$, the elements of $\mathbf{b}$ with indices from $k - a(k)$ to $k + a(k)$ must all be zero as well. $\square$

We now show that every index is contained in either an $\mathbf{a}$- or a $\mathbf{b}$-segment. Since at least one index is contained in both, the conclusion follows.

Assume, to the contrary, that $a(i)$ and $b(i)$ are both positive for some index i; call such indices *bad*. Among all bad indices i, choose one maximising $\max(a(i), b(i))$; by symmetry, we may and will assume that this maximum is $a(i)$. We may and will also assume that either the index $i-1$ is not bad, or $a(i-1) < a(i)$ (otherwise change i to $i-1$, repeat if necessary, recalling that $\mathbf{a}$ is not constant).

Consider the range of indices $\Delta = [i - b(i), i + b(i)]$, and the values $\mathbf{a}$ assumes at those indices. Some indices j in Δ are bad; the corresponding values $a(j)$ do not exceed $a(i)$. Other indices j in Δ are covered by several $\mathbf{a}$- and $\mathbf{b}$-segments. Each $\mathbf{b}$-segment contributes at most $b(i)$ members nearest to one of its endpoints, so the average value of $\mathbf{a}$ over those indices does not exceed $(b(i) - 1)/2 < a(i)$ by Claim 2. The remaining indices j in Δ all lie in $\mathbf{a}$-segments, so the corresponding values $a(j)$ are all zero.

Combining all this, it follows that the average in the right-hand member of $(*)$ does not exceed $a(i)$. Moreover, if some $\mathbf{a}$- or $\mathbf{b}$-segment intersects Δ, then the inequality is strict. Otherwise, $i - 1$ is a bad index contained in Δ, and $a(i-1) < a(i)$, so the inequality is again strict. This contradiction ends the proof and completes the solution.

Solution 2. The solution has a few well-defined steps:

Lemma 1. Assume that $a(i) = M := \max \mathbf{a}$; then $b(i+k) = 0$ for all $k = -M, -M+1, \ldots, M$. In particular, $b(i-1) = b(i) = b(i+1) = 0$, as $M \geq 1$.

Proof. Assume that $a(j) = a(j+1) = \cdots = a(j+s) = M > 0$, and $a(j-1), a(j+s+1) < M$, where $i \in [j, j+s]$. Then $b(j) = 0$, as otherwise $a(j)$ is the mean of at least three terms, all $\leq M$, with at least one $< M$. For the same reason, $b(j+s) = 0$ also.

But then $b(j)$ is the mean of $2M+1$ terms of $\mathbf{b}$, which must therefore all also be equal to 0. So $b(j+k) = 0$ for all $k \in [-M, M]$. Iterating this argument gives $b(j+k) = 0$ for all $k \in [-M, M+s]$. which implies the statement of the lemma. □

Corollary. There exist i such that $a(i) = 0$ and j such that $b(j) = 0$. □

Lemma 2. Suppose $\max \mathbf{a} \geq \max \mathbf{b}$. Generate $\mathbf{a}'$ by replacing all copies of $M = \max \mathbf{a}$ with 1 in $\mathbf{a}$. Then $\mathbf{a}'$ is $\mathbf{b}$-harmonic, and $\mathbf{b}$ is $\mathbf{a}'$-harmonic.

Proof. We start with another consequence of Lemma 1. Assume that $a(i) \neq M$; then none of the terms $a(i+k)$ with $k \in [-b(i), b(i)]$ equals M. Indeed, if $a(i+k) = M$ with $|k| \leq b(i) \leq M$, then by Lemma 1 we have $b(i) = b\big((i+k) - k\big) = 0$, which yields $k = 0$ and hence $a(i) = a(i+k) = M$.

We can now check that the harmonic properties are preserved under replacing all copies of M in $\mathbf{a}$ with 1:

If $a(i) \neq M$, then the harmonic property for $b(i)$ is unchanged. If $a(i) = M$, then $a'(i) = 1$ and $b(i-1) = b(i) = b(i+1) = 0$, so $b(i)$ certainly has the $a'(i)$-harmonic property; and

If $a(i) = M$, then $b(i) = 0$, and so $a'(i) = 1$ has the $b(i)$-harmonic property. If $a(i) \neq M$, then we have just shown that none of the terms in the statement of $a(i)$'s harmonic property are changed by this process, so it remains harmonic.

This check completes the proof of the lemma. □

Lemma 3. We have $\min(a(i), b(i)) = 0$ for all i. Moreover, there exists an i with $a(i) = b(i) = 0$.

Proof. Both statements in the lemma are invariant under the procedure in Lemma 2. Apply this procedure repeatedly, to replace all instances of the maximum value in one of the sequences with 1, until both sequences consist of zeroes and ones. It suffices to check the lemma statement for the obtained pair of sequences.

Suppose that $a(i) = b(i) = 1$ for some i. Since the sequences remain non-constant, we may and will assume that $\min(a(i-1), b(i-1)) = 0$, say $a(i-1) = 0$. But then the $b(i)$-harmonic property is violated for $a(i)$, as $a(i+1) \leq 1$.

Suppose now that there is no i with $a(i) = b(i) = 0$. This means that for every index i we have either $a(i) = 1$ and $b(i) = 0$, or $a(i) = 0$ and $b(i) = 1$. There is a pair of adjacent indices having different types, so that $a(i) = b(i+1) = 1$ and $a(i+1) = b(i) = 0$. But then $b(i)$ violates the $a(i)$-harmonic property. □

Lemma 3 readily yields that at least $N+1$ terms across both sequences are zeroes, as required.

Remark. It can be shown that there are at least $N+2$ zeroes across both sequences, a bound achieved if, for instance, $\mathbf{a} = (0, 0, 1, 1, \ldots, 1, 0)$ and $\mathbf{b} = (1, 0, 0, \ldots, 0)$.

Problem 3. In a country there are n airports and n air companies operating return flights. Each company operates an odd number of flights forming a closed route. Prove that a traveller can complete a closed route consisting of an odd number of flights operated by pairwise distinct companies.

Israel, Ron Aharoni

Solution. In graph-theoretic setting, the statement reads:

> Consider a collection of n odd cycles, not necessarily distinct, all on the same vertex set of size n. Prove that at most one edge can be chosen from each of these cycles to form a collection that contains the edges of an odd cycle.

Call a set of edges *rainbow* if it is formed by choosing at most one edge from each cycle. We have to prove that there exists a rainbow cycle of odd length.

Begin by choosing a maximal rainbow forest F, that is, an acyclic rainbow set of edges.

Since F is acyclic, its size is less than n, so there is a cycle C in the collection no edge of which lies in F. The forest F contains every vertex of C, for otherwise an edge of C incident with a vertex outside F could be added to F to form a larger rainbow forest, contradicting maximality. Moreover, no edge of C joins different components of F, for one such could again be added to F to contradict maximality.

Consequently, the vertices of C all lie in some component of F, a tree T. As such, T is bipartite, that is, its vertices split into two disjoint parts, and all edges are between the two. Since C is an odd cycle, it has an edge whose endpoints both lie in the same part. This edge then completes an odd rainbow cycle.

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Day 2 — Solutions

Problem 4. Let $\mathbb{N}$ be the set of all positive integers. A subset A of $\mathbb{N}$ is *sum-free* if, whenever x and y are (not necessarily distinct) members of A, their sum $x + y$ does not belong to A. Determine all *surjective* functions $f\colon \mathbb{N} \to \mathbb{N}$ such that, for each sum-free subset A of $\mathbb{N}$, the image $\{f(a)\colon a \in A\}$ is again sum-free.

India, Sutanay Bhattacharya

Solution. The identity is the only surjection of the positive integers onto themselves sending every sum-free set onto a sum-free set (no verification is needed, of course).

To prove this, fix a function f satisfying the conditions in the statement, and proceed in several steps.

Step 1. Notice that a 2-element set $\{x, y\}$, where $x < y$, is *not* sum-free if and only if $y = 2x$.

Choose any $a \in \mathbb{N}$, and for any $i \geq 0$ choose some x_i such that $f(x_i) = 2^i a$. The set $f(\{x_i, x_{i+1}\})$ is not sum-free, so neither is $\{x_i, x_{i+1}\}$, whence $x_i = 2x_{i+1}$ or $x_{i+1} = 2x_i$. Since the x_i are all distinct, the same option should hold for all i. The former option yields $x_i = x_0 2^{-i}$ which cannot hold for large enough i. So $x_{i+1} = 2x_i$ for all i.

Therefore, $f(2x) = 2f(x)$ for all x, and, moreover, x is the only argument t with $f(t) = f(2x)/2$. Therefore, f is injective (and hence bijective).

Step 2. Say that a 3-element set $\{a, b, c\}$ is *good* if it is not sum-free, but each of its 2-element subsets is (in other words, no element is twice another). It is easily seen that a set $\{a, b, c\}$, where $a < b$, is good only if $c = b \pm a$. Notice that the pre-image of a good set is also a good set, due to Step 1.

Now let $f(1) = a$. We show that $f(n) = an$ by induction on n. The base cases are $n = 1, 2, 3, 4, 5$; for $n = 1, 2, 4$ the result follows from Step 1.

Set $t = f^{-1}(3a)$ and $s = f^{-1}(5a)$. The sets $\{a, 4a, 3a\}$ and $\{a, 4a, 5a\}$ are good, hence so are $\{1, 4, t\}$ and $\{1, 4, s\}$. Therefore, $\{s, t\} = \{3, 5\}$. But the set $\{a, 5a, 6a\}$ is also good, so the pair $\{1, s\}$ is contained in one more good set, which is not the case if $s = 3$, since $\{1, 3\}$ is contained in one single good set, namely, $\{1, 4, 3\}$. Thus $t = 3$ and $s = 5$, which establishes the base.

For the induction step, assume that $f(k) = ak$ for all $k \leq n$, where $n \geq 5$. Choose $t = f^{-1}((n+1)a)$. Then the pair $\{a, na\}$ is contained in two good sets, namely, $\{a, na, (n-1)a\}$ and $\{a, na, (n+1)a\}$. Their pre-images, $\{1, n, n-1\}$ and $\{1, n, t\}$, are also good, and injectivity of f forces $t = n + 1$. This completes the induction step.

Finally, since f is surjective, $1 = f(n) = an$ for some positive integer n, so $a = 1 = n$. Consequently, f is the identity, as claimed at the beginning of the solution.

Problem 5. A *lattice point* in the Cartesian plane is a point whose coordinates are both integral. A *lattice polygon* is a polygon whose vertices are lattice points. Let Γ be a convex lattice polygon. Prove that Γ is contained in a convex lattice polygon Δ exactly one vertex of which is not a vertex of Γ, and the vertices of Γ all lie on the boundary of Δ.

Russia, Maxim Didin

Solution 1. Let T be the extra vertex of a desired polygon Δ; then Δ is the convex hull of T and Γ. Thus, a point T fits the bill if and only if this convex hull contains no vertices of Γ in its interior.

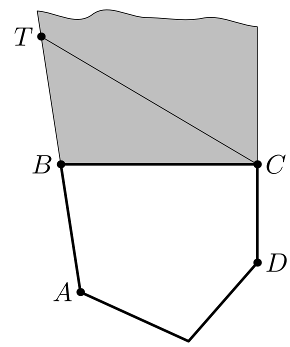
Each segment AB joining two lattice points is partitioned by lattice points into congruent *elementary segments*. Define the *elementary length* $\ell(AB)$ of AB to be the length of any of those elementary segments.

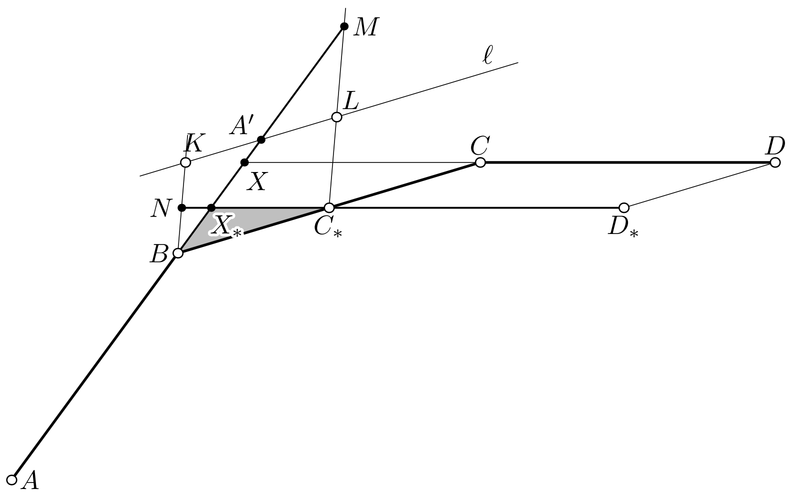
Take any three consecutive sides AB, BC, and CD on the boundary of Γ. Consider the convex region bounded by the segment BC and the rays complementary to the rays BA and CD (including the boundary). If this region is unbounded (see the left figure below), then it contains some lattice point T (e.g., the point with $\overrightarrow{BT} = \overrightarrow{AB}$), and any such point T satisfies the problem requirements. Thus, in what follows we assume that the two rays cross each other at some point X. Assume further that the triangle BCX contains no lattice points outside the segment BC, as any other such point would satisfy the requirements.

Let C_* be a lattice point on the segment BC such that $BC_* = \ell(BC)$, let X_* be the point on BX such that $C_*X_* \parallel CX$, and let D_* be the point such that $\overrightarrow{C_*D_*} = \overrightarrow{CD}$ (see the right figure below). Then the triangle BC_*X_* contains no lattice points apart from B and C_*.

Consider the half-plane determined by the line BC and containing no interior points of Γ. Let ℓ be the line in that half-plane parallel to BC, containing some lattice points, and nearest to BC among such. Let the ray AB meet ℓ at a point A' which belongs to the elementary segment KL on ℓ (we assume that $\overrightarrow{KL} = \overrightarrow{BC_*}$; the point A' may coincide with L but not with K). Then the ray D_*C_* crosses the ray LK (excluding L), otherwise L lies in the triangle BC_*X_*.

The only lattice points contained in the parallelogram $BKLC_*$ are its vertices. This yields that there are no lattice points strictly indise the strip defined by the parallel lines BK and C_*L.





Let M and N be the meeting points of the rays BX_*, C_*L and C_*X_*, BK, respectively. Then the segments BM and C_*N contain no lattice points except their endpoints, so $\ell(AB) \geq BM$ and $\ell(DC) = \ell(D_*C_*) \geq C_*N$. Therefore,

$$\frac{BX_*}{\ell(AB)} + \frac{C_*X_*}{\ell(CD)} \leq \frac{BX_*}{BM} + \frac{C_*X_*}{C_*N} = \frac{BX_*}{BM} + \frac{MX_*}{BM} = 1. \qquad (*)$$

Choose now BC to be a side of largest elementary length. Then

$$(1 \geq)\frac{BX_*}{\ell(AB)} + \frac{C_*X_*}{\ell(CD)} \geq \frac{BX_* + C_*X_*}{\ell(BC)} = \frac{BX_* + C_*X_*}{BC_*},$$

which contradicts the triangle inequality.

Comments. **(1)** The usage of elementary length seems to be crucial. In particular, under the assumption that the triangle BCX contains no lattice points outside BC, an inequality

$$\frac{BX}{AB} + \frac{CX}{CD} \leq 1$$

similar to $(*)$ does not necessarily hold.

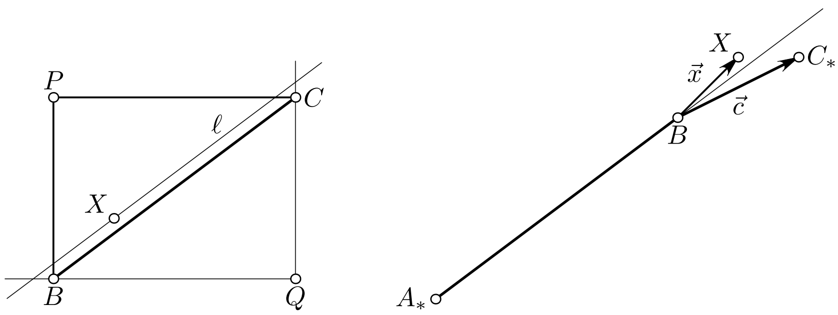
(2) Any lattice parallelogram of unit area may be transformed into a unit square by an affine transform preserving the set of lattice points. If one applies such a transform to the parallelogram $BKLC_*$, inequality $(*)$ may become more transparent.

The key inequality $(*)$ is preserved by affine transforms preserving the lattice, so the above solution is in some sense "affine". In contrast, the next solution, although involving similar ideas, is more "Euclidean".

Solution 2. We use the same notions of *elementary segments* and *elementary lengths* as in Solution 1. We say that a segment is *sloped* if it is neither horizontal nor vertical. If Γ has no sloped sides, then Γ is a rectangle, and the problem statement is trivial: one may choose any lattice point on the extension of any side. So, in what follows, we assume that Γ has a sloped side.

Lemma 1. Let BC be a sloped segment of the boundary of Γ with no lattice points in its interior, and let P be a lattice point such that the segments PB and PC are not sloped, and the triangle PBC lies outside Γ. Then there exists a unique lattice point X in the triangle PBC such that the area of the triangle XBC is $1/2$.

Proof. Choose a lattice point Q such that $PBQC$ is a rectangle. As in Solution 1, let ℓ denote the line through some lattice point parallel to BC, lying outside Γ, and nearest to the line BC under these constraints (see the left figure below). The line ℓ crosses the interior of the angle BQC along an interval of length $> BC$, so this interval should contain a lattice point X. The triangle XBC contains no lattice points apart from the vertices, so its area is $1/2$ due to Pick's formula. Moreover, any such point X should lie within the angle BPC, and ℓ crosses this angle along a segment of length $< BC$. Hence X is the required unique lattice point. $\square$



Denote the point X defined in Lemma 1 by $f(AB)$.

Lemma 2. Let AB and BC be two consecutive sides on the boundary of Γ, and let BA_* and BC_* be elementary segments on the sides BA and BC, respectively. Assume that both coordinates of $\overrightarrow{BC}$ are positive, and that the line AB strictly separates the points C and $X = f(BC_*)$. Then both coordinates of the vector $\overrightarrow{A_*B}$ are also positive, and $BA_* > BC_*$.

Proof. Since AB separates X and C, the vector $\overrightarrow{A_*B}$ is a linear combination of $\vec{c} = \overrightarrow{BC_*}$ and $\vec{x} = \overrightarrow{BX}$ with positive coefficients; so the coordinates of $\overrightarrow{A_*B}$ are positive. Since the area of the triangle BXC equals $1/2$, the vectors $\vec{c}$ and $\vec{x}$ span the whole lattice, so the coefficients of the linear combination are integers. Finally, the angle between $\vec{c}$ and $\vec{x}$ is acute, so $BA_* = |\overrightarrow{A_*B}| \geq \sqrt{|\vec{c}|^2 + |\vec{x}|^2} > |\vec{c}| = BC_*$, as desired. $\square$

Now, choose a sloped side BC of Γ of a maximal elementary length, and let $ABCD$ be the corresponding part of the boundary of Γ. Let C_* and B_* be the points on the segment BC such that $BC_* = B_*C = \ell(BC)$. Let $X = f(BC_*)$ and $X' = f(B_*C)$. Then, due to Lemma 2, the line AB does not separate X and C, and the line CD does not separate X' and B. Therefore, the segment XX' lies in the same angle of the lines AB and CD as Γ, so X may serve as a suitable vertex T of Δ.

Problem 6. For an integer $n > 1$, let $\operatorname{gpf}(n)$ denote the greatest prime factor of n. A *strange pair* is an unordered pair of distinct primes p and q such that $\{p, q\} = \{\operatorname{gpf}(n), \operatorname{gpf}(n+1)\}$ for no integer $n > 1$. Prove that there exist infinitely many strange pairs.

RUSSIA, DMITRY KRACHUN

Solution. We show that there are infinitely many strange pairs of the form $\{2, q\}$ where q is an odd prime.

The Lemma below provides a sufficient condition for such a pair to be strange. For an odd prime q, let $\operatorname{ord}_q(2)$ denote the multiplicative order of 2 modulo q, i.e., the least positive integer s satisfying $q \mid 2^s - 1$.

Lemma. If some primes $2 < q_1 < q_2$ satisfy $\operatorname{ord}_{q_1}(2) = \operatorname{ord}_{q_2}(2)$, then $\{2, q_1\}$ is a strange pair.

Proof. Arguing indirectly, suppose first that $2 = \operatorname{gpf}(n)$ and $q_1 = \operatorname{gpf}(n+1)$; in particular, $n = 2^k$ for some positive integer k, and $q_1 \mid 2^k + 1$. This yields $q_1 \mid 2^{2k} - 1$, so $\operatorname{ord}_{q_2}(2) = \operatorname{ord}_{q_1}(2) \mid 2k$. Therefore, $q_2 \mid 2^{2k} - 1 = (2^k - 1)(2^k + 1)$, but $q_2 \nmid 2^k - 1$, hence $q_2 \mid 2^k + 1$. So $\operatorname{gpf}(n+1) \geq q_2$, which is a contradiction.

Similarly, but easier, if $2 = \operatorname{gpf}(n+1)$ and $q_1 = \operatorname{gpf}(n)$, then $n + 1 = 2^k$, so $\operatorname{ord}_{q_2}(2) = \operatorname{ord}_{q_1}(2) \mid k$ and hence $q_2 \mid 2^k - 1$. Therefore, $\operatorname{gpf}(n+1) \geq q_2$, a contradiction. $\square$

It remains to show that there exist infinitely many disjoint pairs of primes $q_1 < q_2$ satisfying the conditions in the Lemma.

Let $p = 2r - 1 > 5$ be a prime, and let $N = 2^{2p} + 1$. We prove that:

(1) N has at least two distinct prime factors greater than 5; and

(2) $\operatorname{ord}_q(2) = 4p$ for every prime factor $q > 5$ of N.

Thus, every prime $p > 5$ provides a pair of odd primes satisfying the conditions in the Lemma. Moreover, **(2)** shows that distinct primes $p > 5$ provide disjoint such pairs, whence the conclusion.

To prove **(1)**, notice that $3 \nmid N$, and write $N = (4+1) \cdot (4^{p-1} - 4^{p-2} + \cdots + 1) \equiv 5p \pmod{25}$, to infer that $25 \nmid N$.

Next, write $N = (2^p + 1)^2 - 2^{p+1} = (2^p - 2^r + 1)(2^p + 2^r + 1)$. The two factors are coprime (since they are odd, and their difference is 2^{r+1}), and each is larger than 5. Hence each has a prime factor greater than 5. This establishes **(1)**.

To prove **(2)**, consider a prime factor $q > 5$ of N, and notice that $\operatorname{ord}_q(2) \mid 4p$, since $q \mid N \mid 2^{4p} - 1$. If $\operatorname{ord}_q(2) < 4p$, then either $\operatorname{ord}_q(2) \mid 2p$ or $\operatorname{ord}_q(2) \mid 4$. The former is impossible due to $2^{2p} - 1 = N - 2 \equiv -2 \pmod{q}$, the latter — due to $q \nmid 15 = 2^4 - 1$. This establishes **(2)** and completes the proof.