5th Olympiad of Metropolises

Mathematics · Day 1

Problem 1. In a triangle ABC with a right angle at C, the angle bisector AL (where L is on segment BC) intersects the altitude CH at point K. The bisector of angle BCH intersects segment AB at point M. Prove that $CK = ML$.

Problem 2. Does there exist a positive integer n such that all its digits (in the decimal system) are greater than 5, while all the digits of n^2 are less than 5?

Problem 3. Let $n > 1$ be a given integer. The Mint issues coins of n different values a_1, a_2, ..., a_n, where each a_i is a positive integer (the number of coins of each value is unlimited). A set of values $\{a_1, a_2, \dots, a_n\}$ is called *lucky,* if the sum $a_1 + a_2 + \cdots + a_n$ can be collected in a unique way (namely, by taking one coin of each value).

(a) Prove that there exists a lucky set of values $\{a_1, a_2, \dots, a_n\}$ with

$$a_1 + a_2 + \cdots + a_n < n2^n.$$

(b) Prove that every lucky set of values $\{a_1, a_2, \dots, a_n\}$ satisfies

$$a_1 + a_2 + \cdots + a_n > n2^{n-1}.$$

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Mathematics · Day 2

Problem 4. Positive numbers a, b and c satisfy $a^2 = b^2 + bc$ and $b^2 = c^2 + ac$. Prove that $\frac{1}{c} = \frac{1}{a} + \frac{1}{b}$.

Problem 5. There is an empty table with 2^{100} rows and 100 columns. Alice and Eva take turns filling the empty cells of the first row of the table, Alice plays first. In each move, Alice chooses an empty cell and puts a cross in it; Eva in each move chooses an empty cell and puts a zero. When no empty cells remain in the first row, the players move on to the second row, and so on (in each new row Alice plays first).

The game ends when all the rows are filled. Alice wants to make as many different rows in the table as possible, while Eva wants to make as few as possible. How many different rows will be there in the table if both follow their best strategies?

Problem 6. Consider a convex pentagon $ABCDE$. Let A_1, B_1, C_1, D_1, E_1 be the intersection points of the pairs of diagonals BD and CE, CE and DA, DA and EB, EB and AC, AC and BD, respectively. Prove that if four of the five quadrilaterals AB_1A_1B, BC_1B_1C, CD_1C_1D, DE_1D_1E, EA_1E_1A are cyclic, then the fifth one is also cyclic.