

April 2020

Problem 1. The positive integers $a_0, a_1, a_2, \dots, a_{3030}$ satisfy

$$2a_{n+2} = a_{n+1} + 4a_n \text{ for } n = 0, 1, 2, \dots, 3028.$$

Prove that at least one of the numbers $a_0, a_1, a_2, \dots, a_{3030}$ is divisible by 2^{2020} .

Problem 2. Find all lists $(x_1, x_2, \dots, x_{2020})$ of non-negative real numbers such that the following three conditions are all satisfied:

- (i) $x_1 \leq x_2 \leq \dots \leq x_{2020}$;
- (ii) $x_{2020} \leq x_1 + 1$;
- (iii) there is a permutation $(y_1, y_2, \dots, y_{2020})$ of $(x_1, x_2, \dots, x_{2020})$ such that

$$\sum_{i=1}^{2020} ((x_i + 1)(y_i + 1))^2 = 8 \sum_{i=1}^{2020} x_i^3.$$

A permutation of a list is a list of the same length, with the same entries, but the entries are allowed to be in any order. For example, $(2, 1, 2)$ is a permutation of $(1, 2, 2)$, and they are both permutations of $(2, 2, 1)$. Note that any list is a permutation of itself.

Problem 3. Let $ABCDEF$ be a convex hexagon such that $\angle A = \angle C = \angle E$ and $\angle B = \angle D = \angle F$ and the (interior) angle bisectors of $\angle A$, $\angle C$, and $\angle E$ are concurrent.

Prove that the (interior) angle bisectors of $\angle B$, $\angle D$, and $\angle F$ must also be concurrent.

Note that $\angle A = \angle FAB$. The other interior angles of the hexagon are similarly described.

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points

To make this a fair and enjoyable contest for everyone, please do not mention or refer to the problems on the internet or on social media until Saturday 18 April, 22:00 UTC (15:00 Pacific Daylight Time, 23:00 British Summer Time, 08:00 (Sunday) Australian Eastern Standard Time).



Language: English

Day: 2

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Problem 4. A permutation of the integers $1, 2, \dots, m$ is called *fresh* if there exists no positive integer $k < m$ such that the first k numbers in the permutation are $1, 2, \dots, k$ in some order. Let f_m be the number of fresh permutations of the integers $1, 2, \dots, m$.

Prove that $f_n \geq n \cdot f_{n-1}$ for all $n \geq 3$.

For example, if $m = 4$, then the permutation $(3, 1, 4, 2)$ is fresh, whereas the permutation $(2, 3, 1, 4)$ is not.

Problem 5. Consider the triangle ABC with $\angle BCA > 90^\circ$. The circumcircle Γ of ABC has radius R . There is a point P in the interior of the line segment AB such that $PB = PC$ and the length of PA is R . The perpendicular bisector of PB intersects Γ at the points D and E .

Prove that P is the incentre of triangle CDE .

Problem 6. Let $m > 1$ be an integer. A sequence a_1, a_2, a_3, \dots is defined by $a_1 = a_2 = 1$, $a_3 = 4$, and for all $n \geq 4$,

$$a_n = m(a_{n-1} + a_{n-2}) - a_{n-3}.$$

Determine all integers m such that every term of the sequence is a square.

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