The 11th Romanian Master of Mathematics Competition

Day 1: Friday, February 22, 2019, Bucharest

Language: English

Problem 1. Amy and Bob play a game. At the beginning, Amy writes down a positive integer on the board. Then the players alternate moves. Bob moves first. On any move of his, Bob chooses a positive integer b, and replaces the number n on the board with $n - b^2$. On any move of hers, Amy chooses a positive integer k, and replaces the number n on the board with n^k. Bob wins if the number on the board ever becomes zero. Can Amy prevent Bob from winning?

Problem 2. Let $ABCD$ be an isosceles trapezium with AB parallel to DC. Let E be the midpoint of AC. Denote by Γ and Ω the circumcircles of triangles ABE and CDE, respectively. The tangent to Γ at A and the tangent to Ω at D intersect at P. Prove that PE is tangent to Ω.

(The trapezium $ABCD$ with AB parallel to DC is *isosceles* if $\angle BCD = \angle CDA$.)

Problem 3. Given any positive real number ε, prove that for all but finitely many positive integers n, any simple graph on n vertices with at least $(1+\varepsilon)n$ edges has two different simple cycles of equal length.

(A *simple graph* is a set V of vertices, together with a set E of edges, where each edge in E is a set of two vertices of V. A *simple cycle of length k* is a set C of $k \geq 3$ distinct edges in E, such that there is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that for each $1 \leq i < k$, $\{v_i, v_{i+1}\}$ is in C, and $\{v_k, v_1\}$ is in C.)

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.

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Day 2: Saturday, February 23, 2019, Bucharest

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Problem 4. Prove that for every positive integer n there exists a (not necessarily convex) polygon with no three collinear vertices, which admits exactly n different triangulations.

(A *triangulation* is a dissection of the polygon into triangles by interior diagonals which have no common interior points with each other nor with the sides of the polygon.)

Problem 5. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x + yf(x)) + f(xy) = f(x) + f(2019y),$$

for all real numbers x and y.

Problem 6. Find all pairs of integers (c, d), both greater than 1, such that the following holds:

For any monic polynomial Q of degree d with integer coefficients and for any prime $p > c(2c+1)$, there exists a set S of at most $\left(\frac{2c-1}{2c+1}\right)p$ integers, such that

$$\bigcup_{s \in S} \{s,\ Q(s),\ Q(Q(s)),\ Q(Q(Q(s))),\ \ldots\}$$

contains a complete residue system modulo p (i.e., intersects with every residue class modulo p).

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