The 4th Olympiad of Metropolises
Mathematics

Problems

Day 1

Problem 1. Three prime numbers p, q, r and a positive integer n are given such that the numbers

$$\frac{p+n}{qr}, \ \frac{q+n}{rp}, \ \frac{r+n}{pq}$$

are integers. Prove that $p = q = r$. *(Nazar Agakhanov)*

Problem 2. In a social network with a fixed finite set of users, each user has a fixed set of *followers* among the other users. Each user has an initial positive integer rating (not necessarily the same for all users). Every midnight the rating of every user increases by the sum of the ratings that his followers had just before the midnight.

Let m be a positive integer. A hacker, who is not a user of the network, wants all the users to have ratings divisible by m. Every day, he can either choose a user and increase his rating by 1, or do nothing. Prove that the hacker can achieve his goal after some number of days. *(Vladislav Novikov)*

Problem 3. In a non-equilateral triangle ABC point I is the incenter and point O is the circumcenter. A line s through I is perpendicular to IO. Line ℓ symmetric to the line BC with respect to s meets the segments AB and AC at points K and L, respectively (K and L are different from A). Prove that the circumcenter of triangle AKL lies on the line IO. *(Dušan Djukić)*

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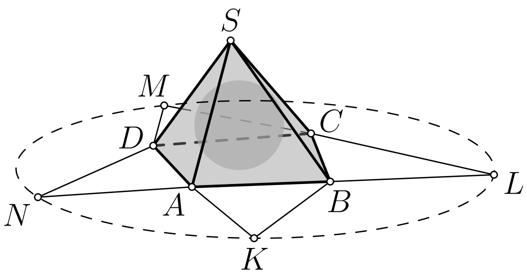
Problem 4. There are 100 students taking an exam. The professor calls them one by one and asks each student a single question: "How many of 100 students will have a "passed" mark by the end of this exam?" The student's answer must be an integer. Upon receiving the answer, the professor immediately publicly announces the student's mark, which is either "passed" or "failed".

After all the students have got their marks, an inspector comes and checks if there is any student who gave the correct answer but got a "failed" mark. If at least one such student exists, then the professor is suspended and all the marks are replaced with "passed". Otherwise no changes are made.

Can the students come up with a strategy that guarantees a "passed" mark to each of them? *(Denis Afrizonov)*

Problem 5. We are given a convex four-sided pyramid with apex S and base face $ABCD$ such that the pyramid has an inscribed sphere (i.e., it contains a sphere which is tangent to each face). By making cuts along the edges SA, SB, SC, SD and rotating the faces SAB, SBC, SCD and SDA outwards into the plane $ABCD$, we unfold the pyramid to the polygon $AKBLCMDN$ as shown in the figure. Prove that points K, L, M, N are concyclic.



(Tibor Bakos and Géza Kós)

Problem 6. Let p be a prime number and let $f(x)$ be a polynomial of degree d with integer coefficients. Assume that the numbers $f(1)$, $f(2)$, $\ldots$, $f(p)$ leave exactly k distinct remainders when divided by p, and $1 < k < p$. Prove that

$$\frac{p-1}{d} \le k-1 \le (p-1)\left(1-\frac{1}{d}\right).$$

(Dániel Domán, Gyula Károlyi and Emil Kiss)