Solutions for 1-st round IZhO 2018

1. Let α, β, γ be the angles of a triangle opposite to the sides a, b, c respectively. Prove the inequality

$$2\left(\cos^2\alpha+\cos^2\beta+\cos^2\gamma\right)\geq\frac{a^2}{b^2+c^2}+\frac{b^2}{a^2+c^2}+\frac{c^2}{a^2+b^2}.$$

Solution. By the Law of Sines, RHS equals $\frac{\sin^2\alpha}{\sin^2\beta+\sin^2\gamma}+\frac{\sin^2\beta}{\sin^2\alpha+\sin^2\gamma}+\frac{\sin^2\gamma}{\sin^2\alpha+\sin^2\beta}$.
Applying Cauchy-Bunyakowski inequality we have

$$\sin^2\alpha=\sin^2(\beta+\gamma)=(\sin\beta\cos\gamma+\sin\gamma\cos\beta)^2\leq(\sin^2\beta+\sin^2\gamma)(\cos^2\gamma+\cos^2\beta),$$

therefore $\cos^2\beta+\cos^2\gamma\geq\frac{\sin^2\alpha}{\sin^2\beta+\sin^2\gamma}$.
Adding similar inequalities for $\cos^2\gamma+\cos^2\alpha$ and $\cos^2\alpha+\cos^2\beta$ we get the desired result.

2. Points N, K, L lie on the sides AB, BC, CA of a triangle ABC respectively so that $AL=BK$ and CN is the bisector of the angle C. The segments AK and BL meet at the point P. Let I and J be the incentres of the triangles APL and BPK respectively. The lines CN and IJ meet at point Q. Prove that $IP=JQ$.

Solution. The case $CA=CB$ is trivial. If $CA\neq CB$, we may suppose, without loss of generality, that CN meets the segment PK.

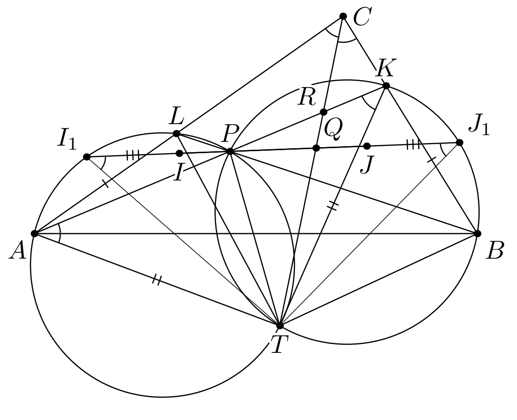

Figure 1: image

Let the circumcircles ω_1 and ω_2 of the triangles APL and BPK respectively meet again at point T. Then

$$\angle LAT=\angle TPB=\angle TKB. \qquad (1)$$

and $\angle ALT=\angle APT=\angle TBK$, that is, $\triangle ALT=\triangle KBT$, hence

$$AT=TK. \qquad (2)$$

It follows from (1) that the quadrilateral $ACKT$ is cyclic; together with (2) this means that $\angle ACT=\angle TCK$, i.e. T lies on the bisector of CN.

Let IJ meet ω_1 and ω_2 at I_1 and J_1 respectively. Since ω_1 and ω_2 have equal radii and $AL=BK$, the triangles ALI_1 and BKJ_1 are equal. We use Mansion's lemma: the midpoint of arc XY of the circumcircle of XYZ lies at equal distances from the ends of this arc and the incentre. It follows from this lemma that $I_1I=I_1L=J_1K=J_1J$. Moreover, $\angle PI_1T=\angle PAT=\angle PKT=\angle PJ_1T$, therefore, $I_1T=J_1T$. Thus T lies on the median bisector of I_1J_1 and on the median bisector of IJ.

It remains to prove that T lies on the median bisector of PQ. Let $R=AK\cap CT$. Then $\angle ART=\angle RAC+\angle ACR=\angle RAC+\angle AKT=\angle RAC+\angle KAT=\angle LAT=\angle BPT$. Since PQ bisects the angle RPB, $\angle PQT=\angle PRT+\angle RPQ=\angle PBT+\angle BPJ=\angle TPQ$, therefore T belongs to the median bisector of PQ and $IP=JQ$.

3. Prove that there exist infinitely many pairs (m,n) of positive integers such that $m+n$ divides $(m!)^n+(n!)^m+1$.
Solution. We shall find a pair such that $m+n=p$ is prime and n is even. Applying Wilson's theorem we have

$$m!=(p-n)!=\frac{(p-1)!}{(p-n+1)\dots(p-2)(p-1)}\equiv\frac{-1}{-(n-1)\dots(-2)(-1)}\equiv\frac{1}{(n-1)!}\equiv\frac{n}{n!}\pmod p.$$

It follows from Fermat's Little Theorem that $(n!)^p\equiv n!\pmod p$, therefore

$$(m!)^n+(n!)^m+1\equiv\left(\frac{n}{n!}\right)^n+(n!)^{p-n}+1\equiv\frac{n^n+n!+(n!)^n}{(n!)^n}\pmod p;$$

thus it suffices to prove that the number $n^n+n!+(n!)^n$ has a prime divisor $p>n$ for infinitely many even n.

We prove that this condition is satisfied, for instance, by all the numbers of the form $n=2q$, where $q>2$ is prime. Let $A=(2q)^{2q}+(2q)!+((2q)!)^{2q}$. For a prime p and integer k we denote by $v_p(k)$ the largest integer ℓ such that p^ℓ divides k.

If $r<2q$ is prime and $r\notin\{2,q\}$ then $A\equiv(2q)^{2q}\not\equiv0\pmod r$. The largest degree of q dividing $(2q)!$ is q^2, while for $(2q)^{2q}$ and $((2q)!)^{2q}$ it is $2q$ and $4q$ respectively, therefore $v_q(A)=2$.

Finally, $v_2((2q)!)=\left[\frac{2q}{2}\right]+\left[\frac{2q}{4}\right]+\left[\frac{2q}{8}\right]+\dots<\frac{2q}{2}+\frac{2q}{4}+\frac{2q}{8}+\dots=2q$, so $v_2((2q)!)<v_2((2q)^{2q})$ and obviously $v_2((2q)!)<v_2((2q)!^{2q})$, thus $v_2(A)\leq 2q-1$. On the other hand, $A>(2q)^{2q}>2^{2q-1}q^2$, therefore A has a prime divisor $p>2q$, q.e.d.

4. The Crocodile thought of four unit squares of a 2018×2018 forming a rectangle with sides 1 and 4. The Bear can choose any square formed by 9 unit squares and ask whether it contains at least one of the four Crocodile's squares. What minimum number of questions should he ask to be sure of at least one affirmative answer?

The answer is $\frac{673^2-1}{2} = 226464$.

Solution. We call *checked* any square chosen by the Bear, and all its unit squares. The position of a unit square in the table can be defined by the numbers of its row and column, that is, the square (x, y) is in the x-th row and y-th column.

First we prove that $\frac{673^2-1}{2}$ questions is enough even on a 2019×2019 table. Let us divide this table into 3×3 squares and apply chess colouring to these large squares so that the corners are white. Thet it is enough to check all the black 3×3 squares: no row or column contains four consecutive white squares.

To prove that we need so many questions, we select all the unit squares with coordinates $(3m + 1, 3n + 1)$, where $0 \leqslant m, n \leqslant 672$. A 3×3 square obviously can not contain two selected unit squares. On the other hand, if two selected squares lie at distance 3 (i.e., one of them is (x, y), and another is $(x, y + 3)$ or $(x + 3, y)$), the Bear must check at least one of these two squares (because if neither is checked, then so are the two unit squares between them, and the Crocodile can place his rectangle on the unchecked squares).

Thus it is enough to produce $\frac{673^2-1}{2}$ pairs of selected unit squares at distance 3. One can take pairs $(6k + 1, 3n + 1)$, $(6k + 4, 3n + 1)$, $0 \leqslant k \leqslant 335$, $0 \leqslant n \leqslant 672$, and $(2017, 6n + 1)$, $(2017, 6n + 4)$, $0 \leqslant n \leqslant 335$.

5. Find all real a for which there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x - f(y)) = f(x) + a[y]$ for every real x и y ($[y]$ denotes the integral part of y).

Answer: $a = -n^2$ for arbitrary integer n.

Solution. First note that $a = 0$ satisfies the problem condition (for example, the equation is satisfied by the function $f(x) \equiv 0$).

Now suppose $a \neq 0$.

Lemma. $f(y) = f(z)$ if and only if $[y] = [z]$.

Suppose $f(y) = f(z)$ for some y, z. Then the given equation implies $f(x)+a[y] = f(x-f(y)) = f(x-f(z)) = f(x)+a[z]$ whence $[y] = [z]$. Conversely, if $[y] = [z]$ then $f(x - f(y)) = f(x) + a[y] = f(x) + a[z = f(x - f(z))]$. It follows from previous observation that $[x - f(y)] = [x - f(z)]$ for all x. Set $x = \frac{f(y)+f(z)}{2}$, then $\left[\frac{f(y)-f(z)}{2}\right] = \left[-\frac{f(y)-f(z)}{2}\right]$, so $f(y) = f(z)$. The lemma is proved.

Now we claim that $f(m) \in \mathbb{Z}$ for any $m \in \mathbb{Z}$. Setting $y = m$ in the given equation we obtain $f(x - f(m)) = f(x) + am$ for any $m \in \mathbb{Z}$, $x \in \mathbb{R}$. Suppose that $f(m) \notin \mathbb{Z}$ for some $m \in \mathbb{Z}$. Choose $t \in (0, 1)$ such that $[f(m)] = [f(m) + t]$. Then for $x = 0$ we have $f(-f(m)) = f(0) + am$ and for $x = -t$ we have $f(-t - f(m)) = f(-t) + am$. Using the lemma we have $f(-f(m)) = f(-t - f(m))$, so $f(0) = f(-t) = f(-1)$, which contradicts the lemma.

From now on we will use in the given equation $f(x - f(y)) = f(x) + ay$ (1) only integer numbers x, y. Setting $y = 1$ in (1) we obtain that $a \in \mathbb{Z}$. Further, for $y = 0$ we have $f(x - f(0)) = f(x)$ and therefore $x - f(0) = x$ (by lemma), whence $f(0) = 0$. Now set $x = f(y)$, then $f(f(y)) = -ay$ (2); replacing y by $f(y)$ in (1) we get $f(x + ay) = f(x) + af(y)$ (3). Denoting $f(1)$ by n and setting $y = 1$ in (3) we obtain $f(x + a) = f(x) + an$ (4). Applying (4) to $x = 0$ we get $f(a) = an$. From (4) we easily conclude that $f(ka) = kan$ for any $k \in \mathbb{Z}$; in particular $f(an) = an^2$. Now setting $y = a$ in (2) gives $-a^2 = f(f(a)) = an^2$ as stated.

It remains to note that if $a = -n^2$ then the function $f(x) = n[x]$ satisfies the given condition: $n[x - n[y]] = n[x] - n^2[y]$, which is obvious.

6. A convex hexagon $ABCDEF$ is inscribed in a circle with radius R. Diagonals AD and BE, BE and CF, AD and CF of the hexagon meet at points M, N, K respectively. Let r_1, r_2, r_3, r_4, r_5, r_6 be the inradii of the triangles ABM, BCN, CDK, DEM, EFN, AFK respectively. Prove that $r_1 + r_2 + r_3 + r_4 + r_5 + r_6 \leqslant R\sqrt{3}$.

Solution.

We start with a lemma.

Lemma. Let R be the circumradius of a quadrilateral $XYZT$, the diagonals of $XYZT$ meet at U, and $\varphi = \frac{1}{2}\angle XUY$. Then the radii r_1 and r_2 of the incentres of XYU and ZTU satisfy

$$\frac{r_1 + r_2}{R} \leqslant 2\tan\varphi(1 - \sin\varphi). \qquad (1)$$

Indeed, let $\angle UXY = 2\psi$, $\angle UYX = 2\vartheta$, then $\angle UTZ = \angle UXY = 2\psi$, $\angle UZT = \angle UYX = 2\vartheta$ (and obviously $\psi + \vartheta + \varphi = \frac{\pi}{2}$). We have $XY + ZT = (r_1 + r_2)(\cot\psi + \cot\vartheta) = 2R\sin\angle XTY + 2R\sin(2\varphi - \angle XTY) = 2R(\sin\angle XTY + \sin(2\varphi - \angle XTY) = 2R \cdot 2\sin\varphi\cos(\varphi - \angle XTY) \leqslant 4R\sin\varphi$. Therefore

$$\frac{r_1 + r_2}{R} \leqslant \frac{4\sin\varphi}{\cot\psi + \cot\vartheta} = \frac{4\sin\varphi\sin\psi\sin\vartheta}{\sin(\psi + \vartheta)} = \frac{4\sin\varphi\sin\psi\sin\vartheta}{\cos\varphi} = 4\tan\varphi\sin\psi\sin\vartheta =$$

$$= 4\tan\varphi \cdot \frac{1}{2}(\cos(\psi - \vartheta) - \cos(\psi + \vartheta)) \leqslant 2\tan\varphi(1 - \sin\varphi),$$

q.e.d.

Returning to the problem, let $\angle AMB = 2\alpha$, $\angle BNC = 2\beta$, $\angle CKD = 2\gamma$, then $\alpha + \beta + \gamma = \frac{\pi}{2}$.

Applying the inequality (1) to the quadrilaterals $ABDE$, $BCEF$ и $CDFA$ we get

$$\frac{r_1 + r_2 + r_3 + r_4 + r_5 + r_6}{R} = \frac{r_1 + r_4}{R} + \frac{r_2 + r_5}{R} + \frac{r_3 + r_6}{R} \leqslant 2\tan\alpha(1 - \sin\alpha) + 2\tan\beta(1 - \sin\beta) + 2\tan\gamma(1 - \sin\gamma).$$

We claim that if $\alpha+\beta+\gamma=\frac{\pi}{2}$ then

$$2\tan\alpha(1-\sin\alpha)+2\tan\beta(1-\sin\beta)+2\tan\gamma(1-\sin\gamma)\leqslant\sqrt{3}. \tag{2}$$

To prove that we consider the function $f(x)=2\tan x(1-\sin x)$ for $x\in(0;\frac{\pi}{2})$.

Since $f''(x)=-2\frac{(1-\sin x)^2+\cos^4 x}{\cos^3 x}<0$ for $x\in(0;\frac{\pi}{2})$, it follows from Jensen's inequality that

$$f(\alpha)+f(\beta)+f(\gamma)\leqslant 3f\left(\frac{\alpha+\beta+\gamma}{3}\right)=3f(\frac{\pi}{6})=\sqrt{3}.$$

Thus (2) is proved, and $r_1+r_2+r_3+r_4+r_5+r_6\leqslant\sqrt{3}R$.