The 10^{th} Romanian Master of Mathematics Competition

Day 1: Friday, February 23, 2018, Bucharest

Language: English

Problem 1. Let $ABCD$ be a cyclic quadrilateral and let P be a point on the side AB. The diagonal AC meets the segment DP at Q. The line through P parallel to CD meets the extension of the side CB beyond B at K. The line through Q parallel to BD meets the extension of the side CB beyond B at L. Prove that the circumcircles of the triangles BKP and CLQ are tangent.

Problem 2. Determine whether there exist non-constant polynomials $P(x)$ and $Q(x)$ with real coefficients satisfying

$$P(x)^{10} + P(x)^9 = Q(x)^{21} + Q(x)^{20}.$$

Problem 3. Ann and Bob play a game on the unit edges of an infinite square grid, making moves in turn. Ann makes the first move. A move consists of orienting any unit edge of the grid that has not been oriented before. If at some stage some oriented edges form an oriented cycle, Bob wins. Does Bob have a strategy that guarantees him to win?

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.

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Day 2: Saturday, February 24, 2018, Bucharest

Language: English

Problem 4. Let a, b, c, d be positive integers such that $ad \neq bc$ and $\gcd(a, b, c, d) = 1$. Let S be the set of values attained by $\gcd(an + b, cn + d)$ as n runs through the positive integers. Show that S is the set of all positive divisors of some positive integer.

Problem 5. Let n be a positive integer and fix $2n$ distinct points on a circle. Determine the number of ways to connect the points with n arrows (i.e. oriented line segments) such that all of the following conditions hold:

- each of the $2n$ points is a startpoint or endpoint of an arrow;
- no two arrows intersect; and
- there are no two arrows $\overrightarrow{AB}$ and $\overrightarrow{CD}$ such that A, B, C and D appear in clockwise order around the circle (not necessarily consecutively).

Problem 6. Fix a circle Γ, a line ℓ tangent to Γ, and another circle Ω disjoint from ℓ such that Γ and Ω lie on opposite sides of ℓ. The tangents to Γ from a variable point X on Ω meet ℓ at Y and Z. Prove that, as X varies over Ω, the circumcircle of XYZ is tangent to two fixed circles.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.