Day 1. Solutions

Problem 1. Solve the system of equations in real numbers:

$$\begin{cases} (x-1)(y-1)(z-1) = xyz - 1\,, \\ (x-2)(y-2)(z-2) = xyz - 2\,. \end{cases}$$

(Vladimir Bragin)

Answer: $x = 1$, $y = 1$, $z = 1$.

Solution 1. By expanding the parentheses and reducing common terms we obtain

$$\begin{cases} -(xy+yz+zx) + (x+y+z) = 0\,, \\ -2(xy+yz+zx) + 4(x+y+z) = 6\,. \end{cases}$$

From the first equation we can conclude that $xy+yz+zx = x+y+z$. By substituting this into the second equation, we obtain that $x+y+z = 3$. We now have to solve the system

$$\begin{cases} x+y+z = 3\,, \\ xy+yz+zx = 3\,. \end{cases} \tag{1}$$

If we square the first equation, we get $x^2+y^2+z^2+2(xy+yz+zx) = 9$. Hence $x^2+y^2+z^2 = 3 = xy+yz+zx$.

We will prove that if $x^2+y^2+z^2 = xy+yz+zx$, than $x = y = z$:

$$x^2+y^2+z^2 = xy+yz+zx \iff$$
$$2x^2+2y^2+2z^2 = 2xy+2yz+2zx \iff$$
$$x^2-2xy+y^2+x^2-2xz+z^2+y^2-2yz+z^2 = 0 \iff$$
$$(x-y)^2+(x-z)^2+(y-z)^2 = 0\,.$$

The sum of three squares is 0, so all of them are zeroes, which implies $x = y = z$. That means $x = y = z = 1$. □

Solution 1'. We will show one more way to solve the system (1). Express $z = 3-x-y$ from first equation and substitute it into the second one:

$$xy+(y+x)(3-x-y) = 3 \iff$$
$$xy+3x+3y-2xy-x^2-y^2 = 3 \iff$$
$$x^2+y^2+xy-3x-3y+3 = 0 \iff$$
$$x^2+x(y-3)+y^2-3y+3 = 0\,.$$

Let us solve it as a quadratic equation over variable x:

$$\begin{aligned} x &= \frac{(3-y) \pm \sqrt{(y-3)^2 - 4(y^2-3y+3)}}{2} = \\ &= \frac{(3-y) \pm \sqrt{y^2-6y+9-4y^2+12y-12}}{2} = \\ &= \frac{(3-y) \pm \sqrt{-3y^2+6y-3}}{2} = \\ &= \frac{(3-y) \pm \sqrt{-3(y-1)^2}}{2}\,. \end{aligned}$$

We can conclude that $y = 1$, because otherwise the square root wouldn't exist. It follows that $x = \frac{3-1\pm 0}{2} = 1$, and then $z = 1$. □

Solution 2. Let's make variable substitution $u = x - 1$, $v = y - 1$, $w = z - 1$. We obtain the system

$$\begin{cases} (u+1)(v+1)(w+1) = uvw + 1\,, \\ (u-1)(v-1)(w-1) = uvw - 1\,, \end{cases}$$

(where the latter equation actually corresponds to the difference between two original equations).

After expanding all parentheses and reducing common terms we have

$$\begin{cases} uv + uw + vw + u + v + w = 0\,, \\ -(uv + uw + vw) + u + v + w = 0\,. \end{cases}$$

By taking the sum and the difference of these equations, we obtain $uv + uw + vu = 0$ and $u + v + w = 0$. Finally, observe that

$$u^2 + v^2 + w^2 = (u+v+w)^2 - 2(uv+uw+vw) = 0 - 0 = 0\,,$$

from which $u = v = w = 0$ follows, and $x = y = z = 1$. □

Solution 3. Consider the polynomial $f(t) = (t-x)(t-y)(t-z)$ with roots x, y, z. We can rewrite the system as

$$\begin{cases} -f(1) = -f(0) - 1\,, \\ -f(2) = -f(0) - 2\,. \end{cases}$$

Now consider the polynomial $g(t) = f(t) - f(0) - t$. Its main coefficient is 1, and 0, 1 and 2 are its roots. Hence $g(t) = t(t-1)(t-2)$. It follows that

$$\begin{aligned} f(t) &= g(t) + t + f(0) = t(t-1)(t-2) + t + f(0) = \\ &= t(t^2-3t+3) + f(0) = t^3 - 3t^2 + 3t - 1 + f(0) + 1 = (t-1)^3 + f(0) + 1\,. \end{aligned}$$

Observe that $(t-1)^3+f(0)+1$ is an increasing function, which means that different real numbers cannot be its roots. So $x=y=z$ and also x is also the root of the derivative of $f(t)$. But $f'(t)=3(t-1)^2$, hence $x=y=z=1$. □

Problem 2. A convex quadrilateral $ABCD$ is circumscribed about a circle ω. Let PQ be the diameter of ω perpendicular to AC. Suppose lines BP and DQ intersect at point X, and lines BQ and DP intersect at point Y. Show that the points X and Y lie on the line AC. *(Géza Kós)*

Solution. The role of points P and Q is symmetrical, so without loss of generality we can assume that P lies inside triangle ACD and Q lies in triangle ABC.

Part 1. Denote the incircles of triangles of ABC and ACD by ω_1 and ω_2 and denote their points of tangency on the diagonal AC by X_1 and X_2, respectively. We will show that line BP passes through X_1, DQ passes through X_2 and $X_1=X_2$. Then it follows that $X=X_1=X_2$ is lying on AC (fig. 1).

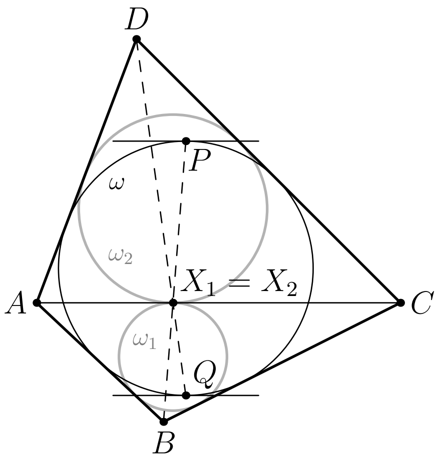

Figure 1: for the solution of the problem 2.

As is well-known, the tangent segments AX_1 and AX_2 to the incircles can be expressed in terms of the side lengths as

$$AX_1=\frac{1}{2}(AB+AC-BC) \quad \text{and} \quad AX_2=\frac{1}{2}(AC+AD-CD).$$

Since the quadrilateral $ABCD$ has an incircle, we have $AB+CD=BC+AD$ and therefore

$$AX_1-AX_2=\frac{1}{2}(AB-BC-AD+CD)=0;$$

this proves $X_1=X_2$.

By having the common tangents BA and BC, the circles ω are ω_1 are homothetic with center B. The tangents to ω at X_1 and to ω_1 at P are parallel, so this homothety maps P to X_1. Hence, the points B, P, X_1 are collinear.

Similarly, from the homothety that maps ω to ω_2, one can see that D, Q, X_2 are collinear.

Part 2. Now let γ_1 and γ_2 be the excircles of triangles of ABC and ACD, opposite to vertices B and D, respectively, and denote their points of tangency on the diagonal AC by Y_1 and Y_2, respectively. Analogously to the first part, we will show that line BQ passes through Y_1, DP passes through Y_2 and $Y_1 = Y_2$ (fig. 2).

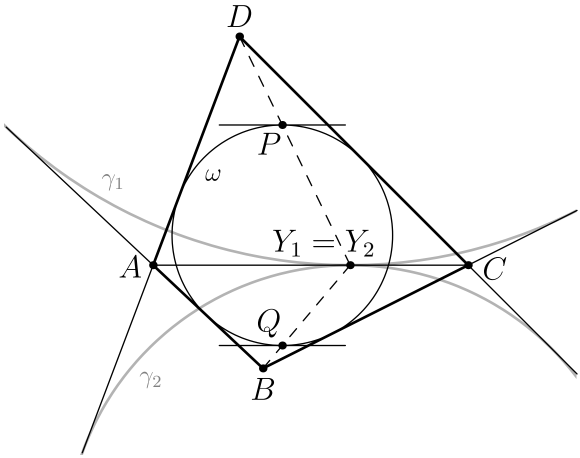

Figure 2: for the solution of the problem 2.

The tangent segments CY_1 and CY_2 to the excircles can be expressed as

$$CY_1 = \frac{1}{2}(AB + AC - BC) \quad \text{and} \quad CY_2 = \frac{1}{2}(AC + AD - CD);$$

by $AB + CD = BC + AD$ it follows that $CY_1 = CY_2$, so $Y_1 = Y_2$.

The circles ω and γ_1 are homothetic with center B. The tangents to ω and γ_1 at Q and Y_1 are parallel so this homothety maps Q to Y_1. Hence, the points B, Q, Y_1 are collinear.

Similarly, from the homothety that maps ω to γ_2, one can see that D, P, Y_2 are collinear. □

Problem 3. Let k be a positive integer such that $p = 8k + 5$ is a prime number. The integers $r_1, r_2, \ldots, r_{2k+1}$ are chosen so that the numbers $0, r_1^4, r_2^4, \ldots, r_{2k+1}^4$ give pairwise different remainders modulo p. Prove that the product

$$\prod_{1\leqslant i<j\leqslant 2k+1}\left(r_i^4+r_j^4\right)$$

is congruent to $(-1)^{k(k+1)/2}$ modulo p.

(Two integers are congruent modulo p if p divides their difference.) *(Fedor Petrov)*

Solution 1. We use the existence of a primitive root g modulo p, that is, such an integer number that the numbers $1, g, g^2, \ldots, g^{p-2}$ give all different non-zero remainders modulo p. Two powers of g, say g^m and g^k, are congruent modulo p if and only if m and k are congruent modulo $p-1$ (the "if" part follows from Fermat's little theorem and the "only if" part from g being primitive root).

There exist exactly $2k+1$ non-zero fourth powers modulo p, namely, $1, g^4, g^8, \ldots, g^{8k}$, thus the numbers $r_1^4, \ldots, r_{2k+1}^4$ are congruent modulo p to them in some order.

Define the map $f(j)\colon \{0,1,\ldots,2k\} \to \{0,1,\ldots,2k\}$ as a remainder of $2j$ modulo $2k+1$. Note that $8j$ and $4f(j)$ are congruent modulo $4(2k+1)=p-1$, therefore $g^{8j} \equiv g^{4f(j)} \pmod p$ for all $j=0,1,\ldots,2k$.

We have

$$\prod_{1\leqslant i<j\leqslant 2k+1}(r_j^4+r_i^4) = \prod_{1\leqslant i<j\leqslant 2k+1}\frac{r_j^8-r_i^8}{r_j^4-r_i^4} \equiv$$

$$\equiv \prod_{0\leqslant i<j\leqslant 2k}\frac{g^{8j}-g^{8i}}{g^{4j}-g^{4i}} \equiv \prod_{0\leqslant i<j\leqslant 2k}\frac{g^{4f(j)}-g^{4f(i)}}{g^{4j}-g^{4i}} \pmod p\,.$$

We may write $g^{4f(j)}-g^{4f(i)} = \pm(g^{4\max(f(j),f(i))}-g^{4\min(f(j),f(i))})$, where the sign is positive if $f(j)>f(i)$ and negative if $f(j)<f(i)$. Further, when the ordered pair (i,j) runs over all $k(2k+1)$ ordered pairs satisfying $0\leqslant i<j\leqslant 2k$, the ordered pair $\big(\min(f(j),f(i)),\max(f(j),f(i))\big)$ runs over the same set. Therefore the differences cancel out and the above product of the ratios $\prod\frac{g^{4f(j)}-g^{4f(i)}}{g^{4j}-g^{4i}}$ equals $(-1)^N$, where N is the number of pairs $i<j$ for which $f(i)>f(j)$. This in turn happens when $i=1,2,\ldots,k$; $j=k+1,\ldots,k+i$, totally $N=1+\ldots+k=k(k+1)/2$. Thus the result. □

Solution 2. Denote $t_i=r_i^4$. Notice that the set $T:=\{t_1,\ldots,t_{2k+1}\}$ consists of distinct roots of the polynomial $x^{2k+1}-1$ (over the field of residues modulo p). Let us re-enumerate T so that $t_{k+1}=1$, $t_i=1/t_{2k+2-i}$ for $i=1,2,\ldots,k$. The map $t\mapsto t^2$ is a bijection on T, the inverse map is $s\mapsto s^{k+1}$ and we naturally denote it $\sqrt{s}$. For distinct elements $t,s\in T$ we have $t+s=\sqrt{st}(\sqrt{s/t}+\sqrt{t/s})$. In the following formula $\prod$ denotes the product over all $k(2k+1)$ pairs of distinct elements

$t, s \in T$. We have

$$\prod(t+s) = \prod \sqrt{st} \cdot \prod(\sqrt{s/t} + \sqrt{t/s}) = \left(\prod_{t\in T} t\right)^k \cdot \left(\prod_{i=1}^{k}(t_i + 1/t_i)\right)^{2k+1}.$$

The first multiple equals 1 by Vieta's formulas for $x^{2k+1} - 1 = \prod_{t\in T}(x-t)$. As for the second multiple, note that there is a polynomial $\psi(x)$ with integer coefficients satisfying

$$\psi\left(x + \frac{1}{x}\right) = x^k + x^{k-1} + \ldots + 1 + \ldots + x^{-k}.$$

Obviously, the leading coefficient in ψ is 1. The constant term can be accessed by substituting the complex unit $x = \mathrm{i}$; the constant term is

$$\psi(0) = \psi\left(\mathrm{i} + \frac{1}{\mathrm{i}}\right) = \sum_{j=-k}^{k} \mathrm{i}^j = \begin{cases} 1 & \text{if } k \equiv 0, 1 \pmod 4, \\ -1 & \text{if } k \equiv 2, 3 \pmod 4. \end{cases}$$

The roots of ψ in the modulo p field are exactly $t_i + 1/t_i$, $i = 1, 2, \ldots, k$ (they are distinct). The product of the roots is

$$\prod_{i=1}^{k}(t_i + 1/t_i) = (-1)^k \cdot \psi(0) = \begin{cases} 1 & \text{if } k \equiv 0, 3 \pmod 4, \\ -1 & \text{if } k \equiv 1, 2 \pmod 4. \end{cases}$$

Finally, we conclude

$$\prod(t+s) = \begin{cases} 1 & \text{if } k \equiv 0, 3 \pmod 4, \\ -1 & \text{if } k \equiv 1, 2 \pmod 4. \end{cases}$$ □

Day 2. Solutions

Problem 4. Let $1 = d_0 < d_1 < \ldots < d_m = 4k$ be all positive divisors of $4k$, where k is a positive integer. Prove that there exists $i \in \{1, \ldots, m\}$ such that $d_i - d_{i-1} = 2$.

(Ivan Mitrofanov)

Solution 1. Assume the contrary. This means that if d and $d+2$ both divide $4k$, then $d+1$ also divides $4k$. Note that if a divides $4k$ and a is not divisible by 4, then $2a$ also divides $4k$. Using the properties above, we start from the pair $(1, 2)$ and find more pairs $(a, a+1)$ such that both a and $a+1$ divide $4k$ and both a and $a+1$ are not divisible by 4.

Let $(a, a+1)$ be a pair of divisors of $4k$ such that 4 divides neither a nor $a+1$. Then $2a$ and $2a+2$ divide $4k$, hence $2a+1$ divides $4k$. One of $2a$ or $2a+2$ is not divisible by 4, hence in one of pairs $(2a, 2a+1)$, $(2a+1, 2a+2)$ both numbers divide $4k$, but are not divisible by 4.

Apply this procedure to the new pair, and so on. Thus, starting from the pair $(1, 2)$ we obtain pairs $(2, 3)$, $(5, 6)$, $(10, 11)$, etc. At each step the sum of numbers in pair increases, hence we obtain an infinite set of divisors of $4k$. A contradiction. □

Solution 2. Assume the contrary. Let t be the minimal positive integer that does not divide $4k$. Then 1, 2, ..., $t-1$ divide $4k$, while t and $t+1$ do not (otherwise $t-1$ and $t+1$ would be two consecutive divisors of $4k$).

It follows that t and $t+1$ are prime powers, otherwise one of them would be a product of two coprime multiples less than t and would therefore divide $4k$. One of them is a power of 2 that we denote as 2^m, $m \geqslant 3$. The other has a form of $2^m + \varepsilon$, $\varepsilon = \pm 1$.

Observe that 2^{m-1} divides $4k$, since 2^m is the minimal even non-divisor of $4k$, and that $3 \cdot 2^{m-2} + \varepsilon$ divides $4k$, since it is odd and less than $2^m + \varepsilon$, which is the minimal odd non-divisor of $4k$. Also note that 3 divides $4k$.

It follows that $3 \cdot 2^{m-1}$ and $2 \cdot (3 \cdot 2^{m-2} + \varepsilon)$ also divide $4k$. Hence, the number $3 \cdot 2^{m-1} + \varepsilon$ between them divides $4k$ too. But now $4k$ is divided by $2 \cdot (3 \cdot 2^{m-1} + \varepsilon)$, just as $4 \cdot (3 \cdot 2^{m-2} + \varepsilon)$. The number $3 \cdot (2^m + \varepsilon)$ is between them, and must be a divisor of $4k$ as well. We conclude that $2^m + \varepsilon$ divides $4k$, contradiction. □

Problem 5. Ann and Max play a game on a 100×100 board.

First, Ann writes an integer from 1 to 10 000 in each square of the board so that each number is used exactly once.

Then Max chooses a square in the leftmost column and places a token on this square. He makes a number of moves in order to reach the rightmost column. In each move the token is moved to a square adjacent by side or by vertex. For each visited square (including the starting one) Max pays Ann the number of coins equal to the number written in that square.

Max wants to pay as little as possible, whereas Ann wants to write the numbers in such a way to maximise the amount she will receive. How much money will Max pay Ann if both players follow their best strategies? *(Lev Shabanov)*

Answer: 500 000 coins.

Solution. Lower bound / Ann's strategy. First we will prove that Ann can get at least 500 000 coins. Suppose Ann has arranged the numbers in the way depicted on the fig. 1.

1	200	201	400	$\cdots$	9800	9801	10000
2	199	202	399	$\cdots$	9799	9802	9999
3	198	203	398	$\cdots$	9798	9803	9998
4	197	204	397	$\cdots$	9797	9804	9997
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
98	103	298	303	$\cdots$	9703	9898	9903
99	102	299	302	$\cdots$	9702	9899	9902
100	101	300	301	$\cdots$	9701	9900	9901

Figure 1: for the solution of problem 5.

Consider a path constructed by Max. For every integer $1 \leqslant n \leqslant 50$ there are two squares of the path in the columns $2n-1$ and $2n$, respectively, which are adjacent by a move of the token. It is easy to see that the sum of the numbers in such squares is at least $200(2n-1)$. We obtain that the cost of this path is at least $200(1+3+5+\ldots+99) = 500\,000$ coins.

Upper bound / Max's strategy. Now consider an arbitrary arrangement of the numbers. Then exclude the square with the greatest number in each column. The least of the numbers in excluded squares is not less than 100, the second least is not less than 200, etc, the greatest number in excluded square is 10 000. Hence, the sum of the excluded numbers is at least $100+200+\ldots+10\,000 = 505\,000$, while the sum of numbers in the whole square is 50 005 000. It means that the sum of the remaining numbers is at most 49 500 000.

The 9900 squares which are left can be split into 99 distinct paths. The first path consists of the lowest squares in each column which are not excluded, the second path consists of the lowest squares in each column which are not excluded and not included into the first path, etc. The last path will include the uppermost squares in each column which are not excluded. It is easy to see that all 99 paths are proper paths that Max's token can follow, because any two squares in two adjacent columns either lie in the same row or in two adjacent rows.

The total cost of the constructed 99 paths is not greater than 49 500 000, therefore one of these paths costs at most 500 000 coins. Thus in every arrangement of numbers Max can pay less than or equal to 500 000 coins.

We proved that Max can pay 500 000 coins or less while Ann can make Max pay at least 500 000 coins, so the answer is 500 000. □

Alternate proof of the upper bound. Split our board into 50 horizontal rectangles 2×100. Since the sum of numbers in the whole board is 50 005 000, it is possible to choose a rectangle with the sum of at most 1 000 100. Take a square with the minimum number in each column of the chosen rectangle. These squares form a path from the square in the left column to the square in the right column; denote the cost of this path as S. In each column of the rectangle, the minimal number is less than the other number by at least 1. Hence, the total sum of the numbers in the rectangle is at least $2S + 100$. It follows that $2S + 100 \leqslant 1\,000\,100$ and $S \leqslant 500\,000$. □

Problem 6. The incircle of a triangle ABC touches the sides BC and AC at points D and E, respectively. Suppose P is the point on the shorter arc DE of the incircle such that $\angle APE = \angle DPB$. The segments AP and BP meet the segment DE at points K and L, respectively. Prove that $2KL = DE$. *(Dušan Djukić)*

A property of the symmedian. The symmedian of a triangle from one of its vertices is defined as the reflection of the median from that vertex about the bisector from the same vertex (fig. 2). In the following solutions we will use a well-known property of the symmedian, namely that it passes through the intersection of the tangents to the circumcircle of the triangle taken at the other two vertices.

Solution 1. Denote by F the tangency point of the incircle with the side AB, and by M and N respectively the midpoints of the segments EF and DF (fig. 3). Since PB is the symmedian in the triangle DPF, we have $\angle KPE = \angle DPB = \angle NPF$. Moreover, $\angle PEK = \angle PED = \angle PFN$, so $\triangle PEK \sim \triangle PFN$. Analogously, $\triangle PDL \sim \triangle PFM$. Now we obtain $EK = FN \cdot \frac{PE}{PF} = \frac{DF \cdot PE}{2PF}$ and similarly $DL = FM \cdot \frac{PD}{PF} = \frac{EF \cdot PD}{2PF}$, so $EK + DL = \frac{DF \cdot PE + EF \cdot PD}{2PF} = \frac{1}{2}DE$ by Ptolemy's theorem. Therefore, $KL = DE - EK - DL = \frac{1}{2}DE$. □

Solution 2. Denote by F the tangency point of the incircle with the side AB. Con-

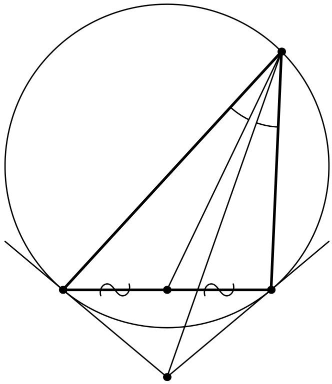

Figure 2: for the solution of problem 6.

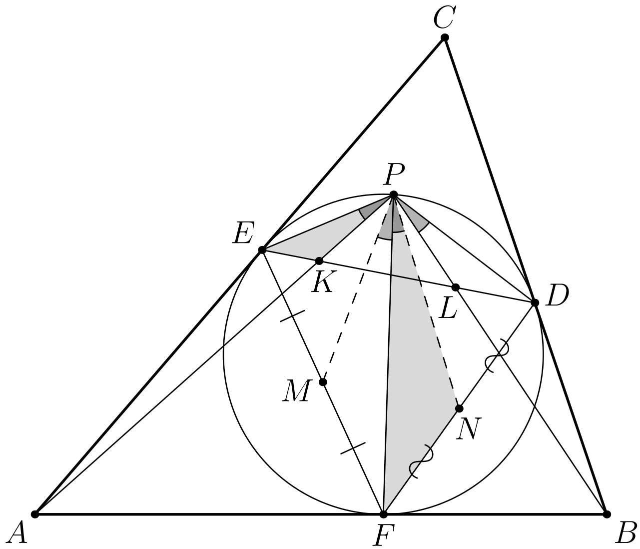

Figure 3: for the solution of problem 6.

sider a point S on the segment ED such that $\angle PSE = \angle PDF$ and $\angle PSD = \angle PEF$ (fig. 4). Triangles PSE and PDF are similar because $\angle PED = \angle PFD$. Since $\angle DPB = \angle KPE$ and PB is a symmedian in triangle PDF, the line PK must be a median in the triangle PSE. It follows that $EK = KS$. Similarly, we have $DL = LS$. Therefore, $2KL = 2KS + 2SL = EK + KS + SL + LD = ED$. □

Solution 3. Again, denote by F the tangency point of the incircle with the side AB. Let the segments AP and BP intersect the incircle again at X and Y, respectively.

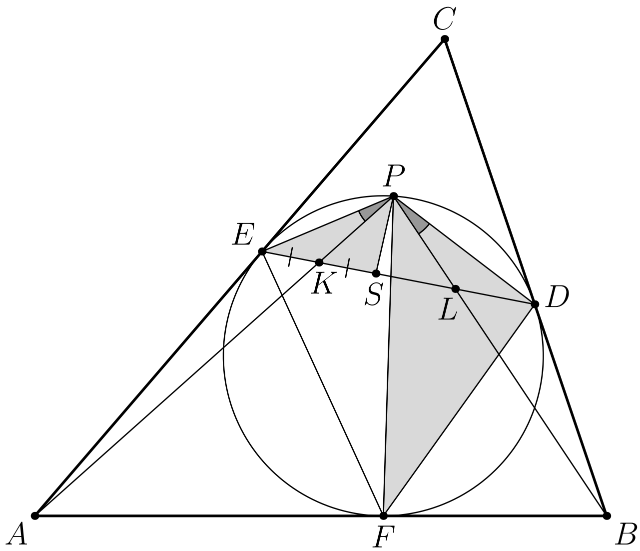

Figure 4: for the solution of problem 6.

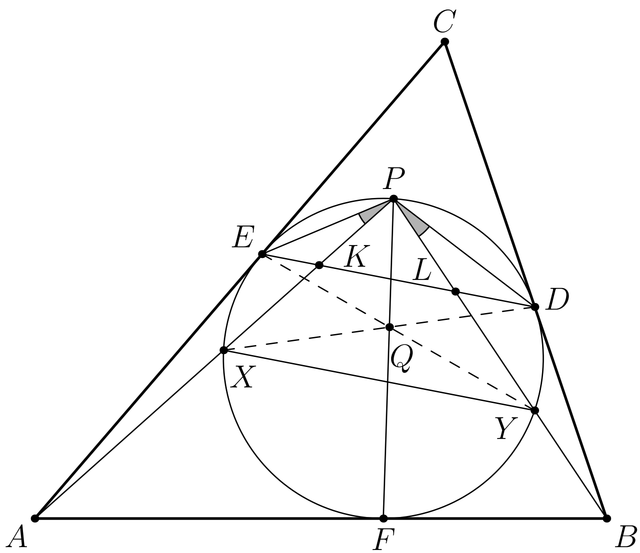

Figure 5: for the solution of the problem 6.

Since the arcs EX and DY are congruent, we have $XY \parallel DE$. Let the lines PF and EY meet at point Q (fig. 5). The quadrilaterals $EPFX$ and $YFPD$ are harmonic, and the projection with the center Q from the incircle to itself maps points E, P, F

to Y, F, P, respectively. Therefore, this projection also sends X to D, i.e. the line XD also passes through Q.

Now let ℓ be the line passing through the intersection point of XE and YD, parallel to DE. Consider the projective transformation preserving the incircle that takes line ℓ to the infinity line. The lines DE, XY and ℓ will remain parallel under this transformation and the ratio KL/DE will not change. Moreover, the quadrilaterals $XEPF$ and $DYFP$ will remain harmonic. Thus we obtain fig. 6.

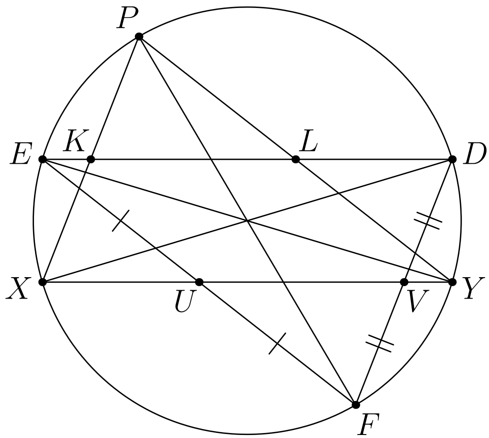

Figure 6: for the solution of the problem 6.

The quadrilateral $XYDE$ is a rectangle and Q is the center, so P and F are diametrically opposite. Suppose the line XY meets the segments EF and DF at points U and V, respectively. Since XP is a symmedian in $\triangle EXF$ and the shorter arcs EP and FY are equal, the line XU is a median in $\triangle EXF$, i.e. $EU = UF$. Analogously, $DV = VF$. It follows that the triangles EFD and UFV are proportional, so $DE/2 = UV = KL$ by symmetry. □