The 3rd Olympiad of Metropolises

Day 1. Problems

Problem 1. Solve the system of equations in real numbers:

$$\begin{cases}(x-1)(y-1)(z-1)=xyz-1\,,\\(x-2)(y-2)(z-2)=xyz-2\,.\end{cases}$$

(Vladimir Bragin)

Problem 2. A convex quadrilateral $ABCD$ is circumscribed about a circle ω. Let PQ be the diameter of ω perpendicular to AC. Suppose lines BP and DQ intersect at point X, and lines BQ and DP intersect at point Y. Show that the points X and Y lie on the line AC. *(Géza Kós)*

Problem 3. Let k be a positive integer such that $p=8k+5$ is a prime number. The integers $r_1, r_2, \ldots, r_{2k+1}$ are chosen so that the numbers $0, r_1^4, r_2^4, \ldots, r_{2k+1}^4$ give pairwise different remainders modulo p. Prove that the product

$$\prod_{1\leqslant i<j\leqslant 2k+1}\left(r_i^4+r_j^4\right)$$

is congruent to $(-1)^{k(k+1)/2}$ modulo p.

(Two integers are congruent modulo p if p divides their difference.) *(Fedor Petrov)*

Day 2. Problems

Problem 4. Let $1 = d_0 < d_1 < \ldots < d_m = 4k$ be all positive divisors of $4k$, where k is a positive integer. Prove that there exists $i \in \{1, \ldots, m\}$ such that $d_i - d_{i-1} = 2$.

(Ivan Mitrofanov)

Problem 5. Ann and Max play a game on a 100×100 board.

First, Ann writes an integer from 1 to 10 000 in each square of the board so that each number is used exactly once.

Then Max chooses a square in the leftmost column and places a token on this square. He makes a number of moves in order to reach the rightmost column. In each move the token is moved to a square adjacent by side or by vertex. For each visited square (including the starting one) Max pays Ann the number of coins equal to the number written in that square.

Max wants to pay as little as possible, whereas Ann wants to write the numbers in such a way to maximise the amount she will receive. How much money will Max pay Ann if both players follow their best strategies? *(Lev Shabanov)*

Problem 6. The incircle of a triangle ABC touches the sides BC and AC at points D and E, respectively. Suppose P is the point on the shorter arc DE of the incircle such that $\angle APE = \angle DPB$. The segments AP and BP meet the segment DE at points K and L, respectively. Prove that $2KL = DE$. *(Dušan Djukić)*