Wednesday, April 11, 2018

Problem 1. Let ABC be a triangle with $CA = CB$ and $\angle ACB = 120^\circ$, and let M be the midpoint of AB. Let P be a variable point on the circumcircle of ABC, and let Q be the point on the segment CP such that $QP = 2QC$. It is given that the line through P and perpendicular to AB intersects the line MQ at a unique point N.

Prove that there exists a fixed circle such that N lies on this circle for all possible positions of P.

Problem 2. Consider the set

$$A = \left\{ 1 + \frac{1}{k} : k = 1, 2, 3, \dots \right\}.$$

(a) Prove that every integer $x \geq 2$ can be written as the product of one or more elements of A, which are not necessarily different.

(b) For every integer $x \geq 2$, let $f(x)$ denote the minimum integer such that x can be written as the product of $f(x)$ elements of A, which are not necessarily different.

Prove that there exist infinitely many pairs (x, y) of integers with $x \geq 2$, $y \geq 2$, and

$$f(xy) < f(x) + f(y).$$

(Pairs (x_1, y_1) and (x_2, y_2) are different if $x_1 \neq x_2$ or $y_1 \neq y_2$.)

Problem 3. The n contestants of an EGMO are named $C_1, \dots, C_n$. After the competition they queue in front of the restaurant according to the following rules.

- The Jury chooses the initial order of the contestants in the queue.
- Every minute, the Jury chooses an integer i with $1 \leq i \leq n$.
  - If contestant C_i has at least i other contestants in front of her, she pays one euro to the Jury and moves forward in the queue by exactly i positions.
  - If contestant C_i has fewer than i other contestants in front of her, the restaurant opens and the process ends.

(a) Prove that the process cannot continue indefinitely, regardless of the Jury's choices.

(b) Determine for every n the maximum number of euros that the Jury can collect by cunningly choosing the initial order and the sequence of moves.

Time: 4 hours and 30 minutes
Each problem is worth 7 points

Thursday, April 12, 2018

Problem 4. A *domino* is a 1×2 or 2×1 tile.

Let $n \geq 3$ be an integer. Dominoes are placed on an $n \times n$ board in such a way that each domino covers exactly two cells of the board, and dominoes do not overlap.

The *value* of a row or column is the number of dominoes that cover at least one cell of this row or column. The configuration is called *balanced* if there exists some $k \geq 1$ such that each row and each column has a value of k.

Prove that a balanced configuration exists for every $n \geq 3$, and find the minimum number of dominoes needed in such a configuration.

Problem 5. Let Γ be the circumcircle of triangle ABC. A circle Ω is tangent to the line segment AB and is tangent to Γ at a point lying on the same side of the line AB as C. The angle bisector of $\angle BCA$ intersects Ω at two different points P and Q.

Prove that $\angle ABP = \angle QBC$.

Problem 6.

(a) Prove that for every real number t such that $0 < t < \frac{1}{2}$ there exists a positive integer n with the following property: for every set S of n positive integers there exist two different elements x and y of S, and a *non-negative* integer m (i.e. $m \geq 0$), such that

$$|x - my| \leq ty.$$

(b) Determine whether for every real number t such that $0 < t < \frac{1}{2}$ there exists an infinite set S of positive integers such that

$$|x - my| > ty$$

for every pair of different elements x and y of S and every *positive* integer m (i.e. $m > 0$).