The 35-th Balkan Mathematical Olympiad

Shortlisted Problems with Solutions

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The shortlisted problems should be kept strictly confidential until the Balkan MO 2019.

Contributing countries

The Organising Committee and the Problem Selection Committee of BMO 2018 thank the following 8 countries for submitting 30 problems in total:

Albania, Bulgaria, Cyprus, Greece, Iran,

FYR Macedonia, Romania, United Kingdom.

Problem Selection Committee

- Dušan Đukić (chairman)
- Marko Radovanović

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P R O B L E M S

Algebra

A1. Let a, b, c be positive real numbers such that $abc = \frac{2}{3}$. Prove that

$$\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a} \geqslant \frac{a+b+c}{a^3+b^3+c^3}.$$

(Dimitar Trenevski, FYR Macedonia)

A2. Two ants start at the same point in the plane. Each minute they choose whether to walk due north, east, south or west. They each walk 1 meter in the first minute. In each subsequent minute the distance they walk is multiplied by a rational number $q > 0$. They meet after a whole number of minutes, but have not taken exactly the same route within that time. Determine all possible values of q. *(Jeremy King, United Kingdom)*

A3. Show that for every positive integer n we have:

$$\sum_{k=0}^{n} \left(\frac{2n+1-k}{k+1}\right)^k = \left(\frac{2n+1}{1}\right)^0 + \left(\frac{2n}{2}\right)^1 + \cdots + \left(\frac{n+1}{n+1}\right)^n \leqslant 2^n.$$

(Dorlir Ahmeti, Albania)

A4. Let a, b, c be positive real numbers such that $abc = 1$. Prove that the following inequality holds:

$$2(a^2+b^2+c^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) \geqslant 3(a+b+c+ab+bc+ca).$$

(Florin Rotaru, Romania)

A5. Let $f : \mathbb{R} \to \mathbb{R}$ be a concave function and let $g : \mathbb{R} \to \mathbb{R}$ be continuous. Given that

$$f(x+y) + f(x-y) - 2f(x) = g(x)y^2$$

for all $x, y \in \mathbb{R}$, prove that f is a quadratic function. *(Peter Gaydarov, Bulgaria)*

A6. Let n be a positive integer and let $x_1, \ldots, x_n$ be real numbers. Show that

$$\sum_{i=1}^{n} x_i^2 \geqslant \frac{1}{n+1}\left(\sum_{i=1}^{n} x_i\right)^2 + \frac{12\left(\sum_{i=1}^{n} ix_i\right)^2}{n(n+1)(n+2)(3n+1)}.$$

(Marios Voskou, Cyprus)

Combinatorics

C1. Let $N \geqslant 3$ be an odd integer. N tennis players take part in a league. Before the league starts, a committee ranks the players in some order based on perceived skill. During the league, each pair of players plays exactly one match, and each match has one winner. A match is considered an *upset* if the winner had a lower initial ranking than the loser. At the end of the league, the players are ranked according to number of wins, with the initial ranking used to rank players with the same number of wins. It turns out that the final ranking is the same as the initial ranking. What is the largest possible number of upsets? *(Dominic Yeo, United Kingdom)*

C2. Alice and Bob play the following game: They start with two non-empty piles of coins. Taking turns, with Alice playing first, they choose a pile with an even number of coins and move half of the coins of this pile to the other pile. The game ends if a player cannot move, or if we reach a previously reached position. In the first case, the player who cannot move loses. In the second case, the game is declared a draw.

Determine all pairs (a, b) of positive integers such that if initially the two piles have a and b coins respectively, then Bob has a winning strategy.

(Demetres Christofides, Cyprus)

C3. An open necklace can contain rubies, emeralds and sapphires. At every step we can perform any of the following operations:

- (1°) We can replace two consecutive rubies with an emerald and a sapphire, where the emerald is on the left of the sapphire.
- (2°) We can replace three consecutive emeralds with a sapphire and a ruby, where the sapphire is on the left of the ruby.
- (3°) If we find two consecutive sapphires then we can remove them.
- (4°) If we find consecutively and in this order a ruby, an emerald, and a sapphire, then we can remove them.

Furthermore we can also reverse all of the above operations. For example, by reversing (3°) we can put two consecutive sapphires on any position we wish.

Initially the necklace has one sapphire (and no other precious stones). Decide, with proof, whether there is a finite sequence of steps such that at the end of this sequence the necklace contains one emerald (and no other precious stones).

<u>*Remark.*</u> A necklace is open if its precious stones are on a line from left to right. We are not allowed to move a precious stone from the rightmost position to the leftmost as we would be able to do if the necklace was closed. *(Demetres Christofides, Cyprus)*

Geometry

G1. In an acute triangle ABC, the midpoint of the side BC is M and the centers of the excircles relative to M of the triangles AMB and AMC are D and E respectively. The circumcircle of the triangle ABD meets line BC at B and F. The circumcircle of the triangle ACE meets line BC at C and G. Prove that $BF = CG$.

(Petru Braica, Romania)

G2. Let ABC be a triangle inscribed in circle Γ with center O and let H its orthocenter and K be the midpoint of OH. The tangent of Γ at B meets the perpendicular bisector of AC meets at L and the tangent of Γ at C meets the perpendicular bisector of AB at M. Prove that $AK \perp LM$. *(Michalis Sarantis, Greece)*

G3. Let P be a point inside a triangle ABC and let a, b, c be the side lengths and p the semi-perimeter of the triangle. Find the maximum value of

$$\min\left(\frac{PA}{p-a}, \frac{PB}{p-b}, \frac{PC}{p-c}\right)$$

over all possible choices of triangle ABC and point P. *(Elton Bojaxhiu, Albania)*

G4. A quadrilateral $ABCD$ is inscribed in a circle k, where $AB > CD$ and AB is not parallel to CD. Point M is the intersection of the diagonals AC and BD and point H is the foot of the perpendicular from M to AB. Given that $\sphericalangle MHC = \sphericalangle MHD$, prove that AB is a diameter of k. *(Emil Stoyanov, Bulgaria)*

G5. Let ABC be an acute-angled triangle with $AB < AC < BC$ and let D be an arbitrary point on the extension of BC beyond C. The circle $c_1(A, AD)$ intersects the rays AC, AB, CB at points E, F, G, respectively. The circumcircle c_2 of triangle AFG intersects the lines FE, BC, GE, DF again at points J, H, H', J'. The circumcircle c_3 of triangle ADE intersects the lines FE, BC, GE, DF again at points I, K, K', I'. Prove that the quadrilaterals $HIJK$ and $H'I'J'K'$ are cyclic and that their circumcenters coincide.

(Vangelis Psychas, Greece)

G6. In a triangle ABC with $AB = AC$, ω is the circumcircle and O its center. Let D be a point on the extension of BA beyond A. The circumcircle ω_1 of triangle OAD intersects the line AC and the circle ω again at points E and G, respectively. Point H is such that $DAEH$ is a parallelogram. Line EH meets circle ω_1 again at point J. The line through G perpendicular to GB meets ω_1 again at point N and the line through G perpendicular to GJ meets ω again at point L. Prove that the points L, N, H, G lie on a circle. *(Theoklitos Paragyiou, Cyprus)*

Number Theory

N1. For positive integers m and n, let $d(m,n)$ be the number of distinct primes that divide both m and n. For instance, $d(60,126)=d(2^2\times 3\times 5, 2\times 3^2\times 7)=2$. Does there exist a sequence (a_n) of positive integers such that:

(i) $a_1 \geqslant 2018^{2018}$;

(ii) $a_m \leqslant a_n$ whenever $m \leqslant n$;

(iii) $d(m,n)=d(a_m,a_n)$ for all positive integers $m\neq n$?

(Dominic Yeo, United Kingdom)

N2. Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that

$$n!+f(m)! \mid f(n)!+f(m!)$$

for all $m,n\in\mathbb{N}$. *(Dorlir Ahmeti and Valmir Krasniqi, Albania)*

N3. Find all primes p and q such that $3p^{q-1}+1$ divides 11^p+17^p.

(Stanislav Dimitrov, Bulgaria)

N4. Let $P(x)=a_dx^d+\cdots+a_1x+a_0$ be a non-constant polynomial with nonnegative integer coefficients having d rational roots. Prove that

$$\operatorname{lcm}\left(P(m),P(m+1),\ldots,P(n)\right)\geqslant m\binom{n}{m}$$

for all positive integers $n>m$. *(Navid Safaei, Iran)*

N5. Let x and y be positive integers. If for each positive integer n we have that

$$(ny)^2+1 \mid x^{\varphi(n)}-1,$$

prove that $x=1$. *(Silouanos Brazitikos, Greece)*

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S O L U T I O N S

Algebra

A1. Let a, b, c be positive real numbers such that $abc = \frac{2}{3}$. Prove that

$$\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a} \geqslant \frac{a+b+c}{a^3+b^3+c^3}.$$

(FYR Macedonia)

Solution.

By the AH mean inequality, we have

$$\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a} = \frac{2}{3(ac+bc)} + \frac{2}{3(ab+ac)} + \frac{2}{3(ab+ac)} \geqslant \frac{3}{ab+ac+bc},$$

so it only remains to prove that $\frac{3}{ab+ac+bc} \geqslant \frac{a+b+c}{a^3+b^3+c^3}$, or equivalently

$$3(a^3+b^3+c^3) \geqslant (a+b+c)(ab+ac+bc).$$

The last inequality easily follows by summing $a^3+b^3 \geqslant ab(a+b)$, $a^3+c^3 \geqslant ac(a+c)$, $b^3+c^3 \geqslant bc(b+c)$ and $a^3+b^3+c^3 \geqslant 3abc$.

A2. Two ants start at the same point in the plane. Each minute they choose whether to walk due north, east, south or west. They each walk 1 meter in the first minute. In each subsequent minute the distance they walk is multiplied by a rational number $q > 0$. They meet after a whole number of minutes, but have not taken exactly the same route within that time. Determine all possible values of q. *(United Kingdom)*

Solution.

Answer: $q = 1$.

Let $x_A^{(n)}$ (resp. $x_B^{(n)}$) be the x-coordinates of the first (resp. second) ant's position after n minutes. Then $x_A^{(n)} - x_A^{(n-1)} \in \{q^n, -q^n, 0\}$, and so $x_A^{(n)}, x_B^{(n)}$ are given by polynomials in q with coefficients in $\{-1, 0, 1\}$. So if the ants meet after n minutes, then

$$0 = x_A^{(n)} - x_B^{(n)} = P(q),$$

where P is a polynomial with degree at most n and coefficients in $\{-2, -, 1, 0, 1, 2\}$. Thus if $q = \frac{a}{b}$ $(a, b \in \mathbb{N})$, we have $a \mid 2$ and $b \mid 2$, i.e. $q \in \{\frac{1}{2}, 1, 2\}$.

It is clearly possible when $q = 1$.

We argue that $q = \frac{1}{2}$ is not possible. Assume that the ants diverge for the first time after the kth minute, for $k \geqslant 0$. Then

$$\left|x_B^{(k+1)} - x_A^{(k+1)}\right| + \left|y_B^{(k+1)} - y_A^{(k+1)}\right| = 2q^k. \tag{1}$$

But also $\left|x_A^{(\ell+1)} - x_A^{(\ell)}\right| + \left|y_A^{(\ell+1)} - y_A^{(\ell)}\right| = q^\ell$ for each $l \geqslant k+1$, and so

$$\left|x_A^{(n)} - x_A^{(k+1)}\right| + \left|y_A^{(n)} - y_A^{(k+1)}\right| \leqslant q^{k+1} + q^{k+2} + \ldots + q^{n-1}. \tag{2}$$

and similarly for the second ant. Combining (1) and (2) with the triangle inequality, we obtain for any $n \geqslant k+1$

$$\left|x_B^{(n)} - x_A^{(n)}\right| + \left|y_B^{(n)} - y_A^{(n)}\right| \geqslant 2q^k - 2\left(q^{k+1} + q^{k+2} + \ldots + q^{n-1}\right),$$

which is strictly positive for $q = \frac{1}{2}$. So for any $n \geqslant k+1$, the ants cannot meet after n minutes. Thus $q \neq \frac{1}{2}$.

Finally, we show that $q = 2$ is also not possible. Suppose to the contrary that there is a pair of routes for $q = 2$, meeting after n minutes. Now consider rescaling the plane by a factor 2^{-n}, and looking at the routes in the opposite direction. This would then be an example for $q = 1/2$ and we have just shown that this is not possible.

Solution 2.

Consider the ants' positions α_k and β_k after k steps in the complex plane, assuming that their initial positions are at the origin and that all steps are parallel to one of the axes. We have $\alpha_{k+1} - \alpha_k = a_k q^k$ and $\beta_{k+1} - \beta_k = b_k q^k$ with $a_k, b_k \in \{1, -1, i, -i\}$.

If $\alpha_n = \beta_n$ for some $n > 0$, then

$$\sum_{k=0}^{n-1} (a_k - b_k) q^k = 0, \quad \text{where} \quad a_k - b_k \in \{0, \pm 1 \pm i, \pm 2, \pm 2i\}.$$

Note that the coefficient $a_k - b_k$ is always divisible by $1+i$ in Gaussian integers: indeed,

$$c_k = \frac{a_k - b_k}{1+i} \in \{0, \pm 1, \pm i, \pm 1 \pm i\}.$$

Canceling $1+i$, we obtain $c_0 + c_1 q + \cdots + c_{n-1} q^{n-1} = 0$. Therefore if $q = \frac{a}{b}$ $(a, b \in \mathbb{N})$, we have $a \mid c_0$ and $b \mid c_{n-1}$ in Gaussian integers, which is only possible if $a = b = 1$.

A3. Show that for every positive integer n we have:

$$\sum_{k=0}^{n}\left(\frac{2n+1-k}{k+1}\right)^k=\left(\frac{2n+1}{1}\right)^0+\left(\frac{2n}{2}\right)^1+\cdots+\left(\frac{n+1}{n+1}\right)^n\leqslant 2^n.$$

(Albania)

Solution.

We shall prove that

$$\binom{n}{k}\geqslant\left(\frac{2n+1-k}{k+1}\right)^k\quad\text{for all}\quad k=0,1,\ldots,n.\qquad(*)$$

The result will follow immediately, as $\sum_{k=0}^{n}\binom{n}{k}=2^n$.

Note that $(*)$ is trivial for $k=0$ and $k=n$. For $0<k<n$, by Hölder's inequality we have

$$\binom{n}{k}=\left(1+\frac{n-k}{k}\right)\cdot\left(1+\frac{n-k}{k-1}\right)\cdots\left(1+\frac{n-k}{1}\right)\geqslant\left(1+\frac{n-k}{\sqrt[k]{k!}}\right)^k.$$

Hence, it is enough to prove that

$$1+\frac{n-k}{\sqrt[k]{k!}}\geqslant\frac{2n+1-k}{k+1}.$$

This is equivalent to $\sqrt[k]{k!}\leqslant\dfrac{k+1}{2}$, which follows from $\sqrt[k]{k!}\leqslant\dfrac{1+2+\cdots+k}{k}=\dfrac{k+1}{2}$.

Solution 2.

As in the previous solution, it is enough to prove $(*)$.

First, we prove that

$$(n-i+1)(n-k+i)(k+1)^2\geqslant i(k-i+1)(2n+1-k)^2\quad\text{for all } i=1,2,\ldots,k.\quad(\sharp)$$

Let us denote the left hand side of the previous inequality with L and the left hand side with R. Then

$$L=(n+1)^2(k+1)^2-(n+1)(k+1)^3+i(k-i+1)(k+1)^2,$$
$$R=4i(k-i+1)(n+1)^2-4i(k-i+1)(n+1)(k+1)+i(k-i+1)(k+1)^2.$$

So, it is enough to prove that

$$(n-k)(k+1)^2\geqslant 4i(k-i+1)(n-k),$$

which follows from

$$(k+1)^2-4i(k-i+1)=(k+1-2i)^2\geqslant 0.$$

Now, by $(\sharp)$ we have

$$\binom{n}{k}^2=\prod_{i=1}^{k}\frac{(n-i+1)(n-k+i)}{i(k-i+1)}\geqslant\prod_{i=1}^{k}\left(\frac{2n+1-k}{k+1}\right)^2=\left(\frac{2n+1-k}{k+1}\right)^{2k},$$

which completes our proof.

A4. Let a, b, c be positive real numbers such that $abc = 1$. Prove that the following inequality holds:

$$2(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \geqslant 3(a+b+c+ab+bc+ca).$$

(Romania)

Solution.

First, we show that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geqslant ab+bc+ca \quad \text{and} \quad \frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geqslant a+b+c. \tag{†}$$

By AG inequality, we have

$$\begin{aligned}\frac{a}{b}+\frac{b}{c}+\frac{c}{a} &= \frac{1}{3}\left(\frac{a}{b}+\frac{a}{b}+\frac{c}{a}\right)+\frac{1}{3}\left(\frac{b}{c}+\frac{b}{c}+\frac{a}{b}\right)+\frac{1}{3}\left(\frac{c}{a}+\frac{c}{a}+\frac{b}{c}\right)\\ &\geqslant \frac{\sqrt[3]{ac}}{\sqrt[3]{b^2}}+\frac{\sqrt[3]{ba}}{\sqrt[3]{c^2}}+\frac{\sqrt[3]{cb}}{\sqrt[3]{a^2}} = \frac{\sqrt[3]{abc}}{b}+\frac{\sqrt[3]{abc}}{c}+\frac{\sqrt[3]{abc}}{a}\\ &= ab+bc+ca.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\frac{a}{b}+\frac{b}{c}+\frac{c}{a} &= \frac{1}{3}\left(\frac{a}{b}+\frac{a}{b}+\frac{b}{c}\right)+\frac{1}{3}\left(\frac{b}{c}+\frac{b}{c}+\frac{c}{a}\right)+\frac{1}{3}\left(\frac{c}{a}+\frac{c}{a}+\frac{a}{b}\right)\\ &\geqslant \frac{\sqrt[3]{a^2}}{\sqrt[3]{bc}}+\frac{\sqrt[3]{b^2}}{\sqrt[3]{ca}}+\frac{\sqrt[3]{c^2}}{\sqrt[3]{ab}} = \frac{a}{\sqrt[3]{abc}}+\frac{b}{\sqrt[3]{abc}}+\frac{c}{\sqrt[3]{abc}}\\ &= a+b+c,\end{aligned}$$

which completes our proof of (†).
By Cauchy-Schwarz inequality we have

$$\left(a^2+b^2+c^2\right)\left(\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{a^2}\right) \geqslant \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2,$$

which together with $(a^2+b^2+c^2)(1/a^2+1/b^2+1/c^2) \geqslant 9$ leads to

$$2\left(a^2+b^2+c^2\right)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \geqslant \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)^2+9 \geqslant 6\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right).$$

Now, the desired inequality follows from (†).

Solution 2.

Set $a = x^3$, $b = y^3$, $c = z^3$ and denote $T_{p,q,r} = \sum_{sym} x^p y^q z^r = x^p y^q z^r + y^p x^q z^r + \cdots$. The given inequality is expanded into

$$4T_{12,6,0} + 2T_{6,6,6} \geqslant 3T_{8,5,5} + 3T_{7,7,4}.$$

Applying the Schur inequality on triples (x^4y^2, y^4z^2, z^4x^2) and (x^2y^4, y^2z^4, z^2x^4) and summing them up yields

$$T_{12,6,0} + T_{6,6,6} \geqslant T_{10,4,4} + T_{8,8,2}. \tag{1}$$

On the other hand, by the Muirhead inequality we have

$$T_{12,6,0} \geqslant T_{6,6,6}, \quad T_{10,4,4} \geqslant T_{8,5,5}, \quad T_{8,8,2} \geqslant T_{7,7,4}. \tag{2}$$

The four inequalities in (1) and (2) imply the desired inequality.

A5. Let $f : \mathbb{R} \to \mathbb{R}$ be a concave function and let $g : \mathbb{R} \to \mathbb{R}$ be continuous. Given that

$$f(x+y) + f(x-y) - 2f(x) = g(x)y^2$$

for all $x, y \in \mathbb{R}$, prove that f is a quadratic function. *(Bulgaria)*

Solution.

We plug in the pairs (a, x), $(a, 2x)$, $(a + x, x)$ and $(a - x, x)$ to get

$$\begin{aligned} f(a+x) + f(a-x) - 2f(a) &= g(a)x^2; && (E1)\\ f(a+2x) + f(a-2x) - 2f(a) &= 4g(a)x^2; && (E2)\\ f(a+2x) + f(a) - 2f(a+x) &= g(a+x)x^2; && (E3)\\ f(a-2x) + f(a) - 2f(a-x) &= g(a-x)x^2, && (E4) \end{aligned}$$

respectively. Combining these equations in the form $2E_1 - E_2 + E_3 + E_4$ the left hand side vanishes, yielding an equation in g: $(g(a+x) + g(a-x) - 2g(a))x^2 = 0$, i.e.

$$g(a) = \frac{g(a+x) + g(a-x)}{2}.$$

Since g is continuous, it must be linear, i.e. $g(x) = c_1 x + c_0$. However, the original equation for $x = y$ together with the concavity condition now gives us

$$0 \geqslant f(2x) + f(0) - 2f(x) = (xc_1 + c_0)x^2$$

for all x, which is only possible if $c_1 = 0$. Thus $g(x) \equiv c_0 = 2A$ is constant and

$$f(x+y) + f(x-y) - 2f(x) = 2Ay^2. \qquad (*)$$

This suggests that f is a quadratic function, so we can set $f(x) = Ax^2 + f_1(x)$. Then $(*)$ becomes $f_1(x+y) + f_1(x-y) - 2f_1(x) = 0$, so an easy induction gives us

$$f_1(nx) - f_1(0) = n(f_1(x) - f_1(0)) \quad \text{for all} \quad n \in \mathbb{Z}.$$

By setting $f_1(0) = C$ and $f_1(1) = B + C$ we obtain $f_1(x) = Bx + C$ and $f(x) = Ax^2 + Bx + C$ for all $x \in \mathbb{Q}$. By concavity of f we conclude that $f(x) = Ax^2 + Bx + C$ for all real x.

Remark.

In fact, $(*)$ implies that the second derivative of f is constant by taking $y \to 0$ and the problem is solved. The solution presented here avoids use of derivatives.

A6. Let n be a positive integer and let $x_1, \ldots, x_n$ be real numbers. Show that

$$\sum_{i=1}^{n} x_i^2 \geqslant \frac{1}{n+1}\left(\sum_{i=1}^{n} x_i\right)^2 + \frac{12\left(\sum_{i=1}^{n} i x_i\right)^2}{n(n+1)(n+2)(3n+1)}.$$

(Cyprus)

Solution.

Let $S = \frac{1}{n+1}\sum_{i=1}^{n} x_i$, and $y_i = x_i - S$ for $1 \leqslant i \leqslant n$. Then we have

$$\sum_{i=1}^{n} i x_i = \sum_{i=1}^{n} i y_i + \frac{n(n+1)}{2} S$$

and

$$\sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} x_i^2 - 2S\sum_{i=1}^{n} x_i + nS^2 = \sum_{i=1}^{n} x_i^2 - \frac{1}{n+1}\left(\sum_{i=1}^{n} x_i\right)^2 - S^2.$$

Now, by the Cauchy-Schwarz inequality

$$\begin{aligned}\left(\sum_{i=1}^{n} i x_i\right)^2 &= \left(\sum_{i=1}^{n} i y_i + \frac{n(n+1)}{2} S\right)^2 \\ &\leqslant \left(\sum_{i=1}^{n} i^2 + \frac{n^2(n+1)^2}{4}\right)\left(\sum_{i=1}^{n} y_i^2 + S^2\right) \\ &= \frac{n(n+1)(n+2)(3n+1)}{12} \cdot \left(\sum_{i=1}^{n} x_i^2 - \frac{1}{n+1}\left(\sum_{i=1}^{n} x_i\right)^2\right),\end{aligned}$$

which completes our proof.

Remark.

It can be checked that equality holds if and only if $x_i = c(n(n+1) + 2i)$ for $1 \leqslant i \leqslant n$ and some $c \in \mathbb{R}$.

Combinatorics

C1. Let $N \geqslant 3$ be an odd integer. N tennis players take part in a league. Before the league starts, a committee ranks the players in some order based on perceived skill. During the league, each pair of players plays exactly one match, and each match has one winner. A match is considered an *upset* if the winner had a lower initial ranking than the loser. At the end of the league, the players are ranked according to number of wins, with the initial ranking used to rank players with the same number of wins. It turns out that the final ranking is the same as the initial ranking. What is the largest possible number of upsets? *(United Kingdom)*

Solution.

Answer: $\dfrac{(N-1)(3N-1)}{8}$.

Suppose the players are ranked $1, 2, \ldots, N = 2n+1$, where 1 is the highest ranking.

For $k \leqslant n$, the player ranked k could have beaten at most $k-1$ players with a higher ranking. Thus the top n players could have made at most $\sum_{k=1}^{n}(k-1) = \frac{n(n-1)}{2}$ upsets. On the other hand, the average score of all $2n+1$ players is n, so the average score of the bottom $n+1$ players is not more than n, which implies that these $n+1$ players have at most $n(n+1)$ wins in total. Hence the total number of upsets is at most

$$\frac{n(n-1)}{2} + n(n+1) = \frac{n(3n+1)}{2} = \frac{(N-1)(3N-1)}{8}.$$

An example can be constructed as follows. Suppose that, for $1 \leqslant i \leqslant 2n+1$, the player ranked i beats the players ranked $i-1, i-2, \ldots, i-n$ (the rankings are counted modulo N) and loses to the rest of the players. Thus each player has exactly n wins. The player ranked i for $i \leqslant n$ made $i-1$ upsets, whereas the player ranked i for $i > n$ made n upsets, so the total number of upsets is exactly $\sum_{i=1}^{n}(i-1) + (n+1)n = \frac{n(3n+1)}{2}$.

Solution 2.

Write $N = 2n+1$. We only prove the upper bound.

Consider a tournament $\mathbb{T}$ with correct final ranking, but where not everyone won n matches. Let A be the worst-ranked player with the maximal number of wins, and let B be the best-ranked player with minimal wins. Clearly, A was ranked above B.

Assume A beat B. Consider the tournament $\mathbb{T}'$ obtained from $\mathbb{T}$ by reversing this result, and keeping all others the same. So B beat A, which is an upset. A is now the best-ranked player with the second-most number of wins; B is now the worst-ranked player with the second-least number of wins, and so the final ranking of $\mathbb{T}'$ is still correct, but with one more upset than in $\mathbb{T}$.

Alternatively, assume B beat A. Then there must have been a player C such that A beat C and C beat B. These are upsets if, respectively, C was ranked above A, or below B. It therefore cannot be the case that both of the matches involving C and $\{A, B\}$ were upsets. Consider the tournament $\mathbb{T}'$ obtained from $\mathbb{T}$ by reversing these two matches. C's number of wins stays fixed, while as before A is now the best-ranked player with the second-most wins, and similar for B. Thus in $\mathbb{T}'$ the final ranking is still correct, with either the same number of upsets as $\mathbb{T}$, or two more upsets than $\mathbb{T}$.

If we iterate this procedure, we eventually obtain a tournament $\bar{\mathbb{T}}$ where everyone won exactly n matches, and with at least as many upsets as in the original tournament $\mathbb{T}$. We now bound the number of upsets in such a tournament $\bar{\mathbb{T}}$. Suppose the player ranked $i \leqslant \frac{N+1}{2}$ beat K higher ranked players. Obviously $K \leqslant i - 1$. Then the number of upsets involving i is

$$2K + \frac{N+1}{2} - i \leqslant 2(i-1) + \frac{N+1}{2} - i = \frac{N-1}{2} + i - 1.$$

Similarly, for $i \geqslant \frac{N+1}{2}$ one proves that the number of upsets involving i is at most $\frac{N-1}{2} + i - 1$.

Finally, summing over all values of i and dividing by 2 we obtain the desired result.

Remark.

We demand N odd to avoid candidates providing a case distinction, rather than because the construction or the bounding argument is significantly different.

C2. Alice and Bob play the following game: They start with two non-empty piles of coins. Taking turns, with Alice playing first, they choose a pile with an even number of coins and move half of the coins of this pile to the other pile. The game ends if a player cannot move, or if we reach a previously reached position. In the first case, the player who cannot move loses. In the second case, the game is declared a draw.

Determine all pairs (a, b) of positive integers such that if initially the two piles have a and b coins respectively, then Bob has a winning strategy. *(Cyprus)*

Solution.

By $v_2(n)$ we denote the largest nonnegative integer r such that $2^r \mid n$.

A position (a, b) (i.e. two piles of sizes a and b) is said to be k-*happy* if $v_2(a) = v_2(b) = k$ for some integer $k \geqslant 0$, and k-unhappy if $\min\{v_2(a), v_2(b)\} = k < \max\{v_2(a), v_2(b)\}$. We shall prove that Bob has a winning strategy if and only if the initial position is k-happy for some even k.

- Given a 0-happy position, the player in turn is unable to play and loses.
- Given a k-happy position (a, b) with $k \geqslant 1$, the player in turn will transform it into one of the positions $(a + \frac{1}{2}b, \frac{1}{2}b)$ and $(b + \frac{1}{2}a, \frac{1}{2}a)$, both of which are $(k-1)$-happy because $v_2(a + \frac{1}{2}b) = v_2(\frac{1}{2}b) = v_2(b + \frac{1}{2}a) = v_2(\frac{1}{2}a) = k - 1$.

Therefore, if the starting position is k-happy, after k moves they will get stuck at a 0-happy position, so Bob will win if and only if k is even.

- Given a k-unhappy position (a, b) with k odd and $v_2(a) = k < v_2(b) = \ell$, Alice can move to position $(\frac{1}{2}a, b + \frac{1}{2}a)$. Since $v_2(\frac{1}{2}a) = v_2(b + \frac{1}{2}a) = k - 1$, this position is $(k-1)$-happy with $2 \mid k - 1$, so Alice will win.
- Given a k-unhappy position (a, b) with k even and $v_2(a) = k < v_2(b) = \ell$, Alice must not play to position $(\frac{1}{2}a, b + \frac{1}{2}a)$, because the new position is $(k-1)$-happy and will lead to Bob's victory. Thus she must play to position $(a + \frac{1}{2}b, \frac{1}{2}b)$. We claim that this position is also k-unhappy. Indeed, if $\ell > k + 1$, then $v_2(a + \frac{1}{2}b) = k < v_2(\frac{1}{2}b) = \ell - 1$, whereas if $\ell = k + 1$, then $v_2(a + \frac{1}{2}b) > v_2(\frac{1}{2}b) = k$.

Hence a k-unhappy position is winning for Alice if k is odd, and drawing if k is even.

C3. An open necklace can contain rubies, emeralds and sapphires. At every step we can perform any of the following operations:

(1°) We can replace two consecutive rubies with an emerald and a sapphire, where the emerald is on the left of the sapphire.

(2°) We can replace three consecutive emeralds with a sapphire and a ruby, where the sapphire is on the left of the ruby.

(3°) If we find two consecutive sapphires then we can remove them.

(4°) If we find consecutively and in this order a ruby, an emerald, and a sapphire, then we can remove them.

Furthermore we can also reverse all of the above operations. For example, by reversing (3°) we can put two consecutive sapphires on any position we wish.

Initially the necklace has one sapphire (and no other precious stones). Decide, with proof, whether there is a finite sequence of steps such that at the end of this sequence the necklace contains one emerald (and no other precious stones).

Remark. A necklace is open if its precious stones are on a line from left to right. We are not allowed to move a precious stone from the rightmost position to the leftmost as we would be able to do if the necklace was closed. *(Cyprus)*

Solution.

For each precious stone on the necklace, we define its value as $(-1)^r \cdot s$, where r denotes the number of emeralds and sapphires preceding it, and s equals -2, 1 or -1 for a ruby, emerald or sapphire, respectively.

The value of the necklace is equal to the sum of the values of its precious stones. We claim that the value of the necklace is invariant modulo 6.

Suppose for example that we remove two consecutive rubies, and suppose there is an even number of emeralds and sapphires preceding them. The value of each ruby is -2 so by removing them we increase the value of the necklace by 4. The emerald that we add had an even number of emeralds and sapphires preceding it, so its value is 1. The sapphire that we add has an odd number of emeralds and sapphires preceding it (accounting for the added emerald), so its value is 1. No other precious stone changes value, so the total increase of the value of the necklace is 6.

Similarly we can check that all of the other operations and their inverses also leave the value of the necklace invariant modulo 6.

Since the necklace containing just one sapphire has value -1, whereas the necklace containing just one emerald has value 1, there is no desired sequence of steps.

Solution 2.

Write a, b and c respectively for a ruby, emerald and sapphire. Each necklace corresponds to an element of a group G containing elements a, b, c. If we impose the conditions $a^2 = bc$, $b^3 = ca$, $c^2 = 1$ and $abc = 1$, the allowed operations will preserve this element. In this group we have $c = ab$ (since $c^2 = abc$), i.e. $b = a^{-1}c$ and using this relation we obtain $a^3 = c^2 = (a^{-1}c)^4 = 1$. Thus we can take $G = \mathbb{S}_4$, $a = (1, 2, 3)$, $c = (1, 4)$ and $b = a^{-1}c = (1, 4, 3, 2)$. The initial and final necklaces should correspond to elements c and b, respectively, so the desired sequence of operations does not exist.

Geometry

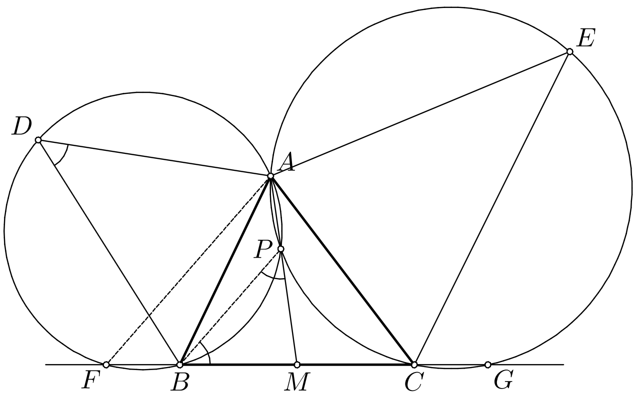
G1. In an acute triangle ABC, the midpoint of the side BC is M and the centers of the excircles relative to M of the triangles AMB and AMC are D and E respectively. The circumcircle of the triangle ABD meets line BC at B and F. The circumcircle of the triangle ACE meets line BC at C and G. Prove that $BF = CG$. *(Romania)*

Solution.

We have $\sphericalangle ADB = 90^\circ - \frac{1}{2}\sphericalangle AMB$ and $\sphericalangle AEC = 90^\circ - \frac{1}{2}\sphericalangle AMC$.

Let the circles ADB and AEC respectively meet the line AM again at points P and P'. Note that M lies outside the circles ABD and ACE because $\sphericalangle ADB + \sphericalangle AMB < 180^\circ$ and $\sphericalangle AEC + \sphericalangle AMC < 180^\circ$, so P and P' lie on the ray MA. Moreover, $\sphericalangle BPM = \sphericalangle BDA = 90^\circ - \frac{1}{2}\sphericalangle PMB$, implying that $\triangle BPM$ is isosceles with $MP = MB$. Similarly, $MP' = MC = MB$, so $P' \equiv P$.

Now it follows from the power of point P that $MB \cdot MF = MP \cdot MA = MC \cdot MG$, i.e. $MF = MG = MA$ and hence $BF = CG$.

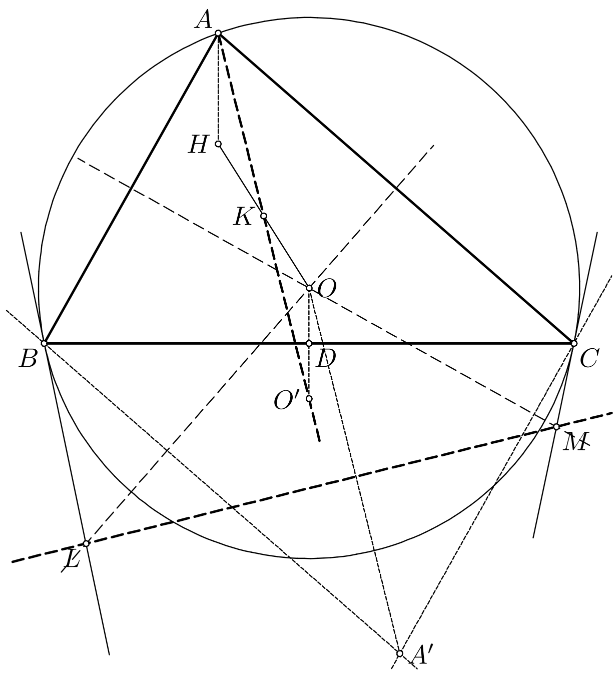


G2. Let ABC be a triangle inscribed in circle Γ with center O and let H its orthocenter and K be the midpoint of OH. The tangent of Γ at B meets the perpendicular bisector of AC meets at L and the tangent of Γ at C meets the perpendicular bisector of AB at M. Prove that $AK \perp LM$. *(Greece)*

Solution.

The polar of L with respect to Γ is the line ℓ_B through B parallel to AC, and the polar of M with respect to Γ is the line ℓ_C through C parallel to AB. Therefore the pole of the line LM is the intersection A' of ℓ_B and ℓ_C. It follows that $OA' \perp LM$, so it remains to show that $OD \parallel AK$.

Consider the reflection O' of O in the midpoint D of BC. Since A' is the reflection of A in D, $AOA'O'$ is a parallelogram. Moreover, $AHO'O$ is a parallelogram because $\overrightarrow{OO'} = 2\overrightarrow{OD} = \overrightarrow{AH}$. It follows that $\overrightarrow{OA'} = \overrightarrow{AO'} = 2\overrightarrow{AK}$, so $OA' \parallel AK$.



Solution 2.

We introduce the complex plane such that Γ is the unit cycle. Also, let the lower-case letters denote complex numbers corresponding to the points denoted by capital letters. First, note that $o = 0$, $\overline{a} = 1/a$, $\overline{b} = 1/b$ and $\overline{c} = 1/c$.

Since $BL \perp BO$, we have

$$\frac{b-l}{\overline{b}-\overline{l}} = -\frac{b-o}{\overline{b}-\overline{o}} = -\frac{b}{\overline{b}} = -b^2, \quad \text{and hence} \quad \overline{l} = \frac{2b-l}{b^2}. \tag{†}$$

Since $LO \perp AC$, we have

$$\frac{l}{\overline{l}} = \frac{l-o}{\overline{l}-\overline{o}} = -\frac{a-c}{\overline{a}-\overline{c}} = ac, \quad \text{and hence} \quad \overline{l} = \frac{l}{ac}. \tag{‡}$$

Combining (†) and (‡) we get $l = \dfrac{2abc}{b^2 + ac}$. By symmetry, $m = \dfrac{2abc}{c^2 + ab}$ and hence

$$l - m = \frac{2abc(c-b)(b+c-a)}{(b^2+ac)(c^2+ab)} \quad \text{and} \quad \overline{l} - \overline{m} = \frac{2(b-c)(ab+ac-bc)}{(b^2+ac)(c^2+ab)}.$$

By Hamilton's formula $a + b + c = h - o = h$, and hence $k = \dfrac{h+o}{2} = \dfrac{a+b+c}{2}$.
So,

$$a - k = \frac{b+c-a}{2} \quad \text{and} \quad \overline{a} - \overline{k} = \frac{ab+ac-bc}{2abc},$$

and hence

$$\frac{l-m}{\overline{l}-\overline{m}} = -\frac{a-k}{\overline{a}-\overline{k}},$$

which implies $LM \perp AK$.

G3. Let P be a point inside a triangle ABC and let a, b, c be the side lengths and p the semi-perimeter of the triangle. Find the maximum value of

$$\min\left(\frac{PA}{p-a}, \frac{PB}{p-b}, \frac{PC}{p-c}\right)$$

over all possible choices of triangle ABC and point P. *(Albania)*

Solution.

If ABC is an equilateral triangle and P its center, then $\frac{PA}{p-a} = \frac{PB}{p-b} = \frac{PC}{p-c} = \frac{2}{\sqrt{3}}$.

We shall prove that $\frac{2}{\sqrt{3}}$ is the required value. Suppose without loss of generality that $\sphericalangle APB \geqslant 120^\circ$. Then

$$AB^2 \geqslant PA^2 + PB^2 + PA \cdot PB \geqslant \frac{3}{4}(PA + PB)^2,$$

i.e. $PA + PB \leqslant \frac{2}{\sqrt{3}}AB = \frac{2}{\sqrt{3}}((p-a) + (p-b))$, so at least one of the ratios $\frac{PA}{p-a}$ and $\frac{PB}{p-b}$ does not exceed $\frac{2}{\sqrt{3}}$.

G4. A quadrilateral $ABCD$ is inscribed in a circle k, where $AB > CD$ and AB is not parallel to CD. Point M is the intersection of the diagonals AC and BD and point H is the foot of the perpendicular from M to AB. Given that $\sphericalangle MHC = \sphericalangle MHD$, prove that AB is a diameter of k. *(Bulgaria)*

Solution.

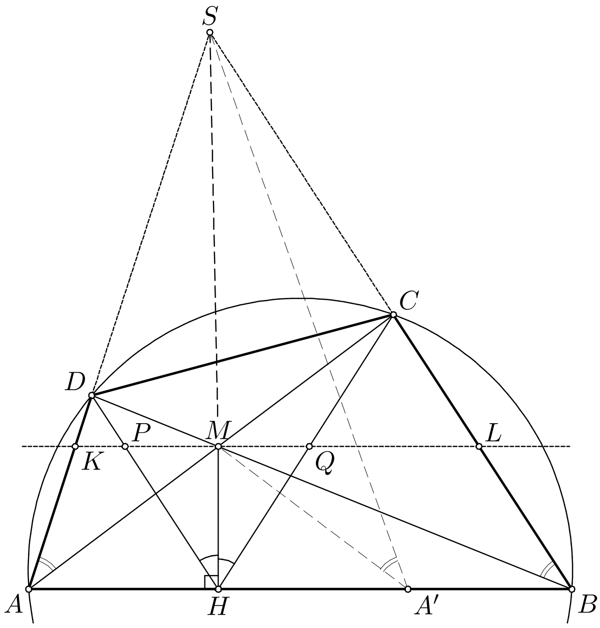
Let the line through M parallel to AB meet the segments AD, DH, BC, CH at points K, P, L, Q, respectively. Triangle HPQ is isosceles, so $MP = MQ$. Now from

$$\frac{MP}{BH} = \frac{DM}{DB} = \frac{KM}{AB} \quad \text{and} \quad \frac{MQ}{AH} = \frac{CM}{CA} = \frac{ML}{AB}$$

we obtain $AH/HB = KM/ML$.

Let the lines AD and BC meet at point S and let the line SM meet AB at H'. Then $AH'/H'B = KM/ML = AH/HB$, so $H' \equiv H$, i.e. S lies on the line MH.

The quadrilateral $ABCD$ is not a trapezoid, so $AH \neq BH$. Consider the point A' on the ray HB such that $HA' = HA$. Since $\sphericalangle SA'M = \sphericalangle SAM = \sphericalangle SBM$, quadrilateral $A'BSM$ is cyclic and therefore $\sphericalangle ABC = \sphericalangle A'BS = \sphericalangle A'MH = \sphericalangle AMH = 90° - \sphericalangle BAC$, which implies that $\sphericalangle ACB = 90°$.



G5. Let ABC be an acute-angled triangle with $AB < AC < BC$ and let D be an arbitrary point on the extension of BC beyond C. The circle $\gamma(A, AD)$ intersects the rays AC, AB, CB at points E, F, G, respectively. The circumcircle ω_1 of triangle AFG intersects the lines FE, BC, GE, DF again at points J, H, H', J'. The circumcircle ω_2 of triangle ADE intersects the lines FE, BC, GE, DF again at points I, K, K', I'. Prove that the quadrilaterals $HIJK$ and $H'I'J'K'$ are cyclic and that their circumcenters coincide.

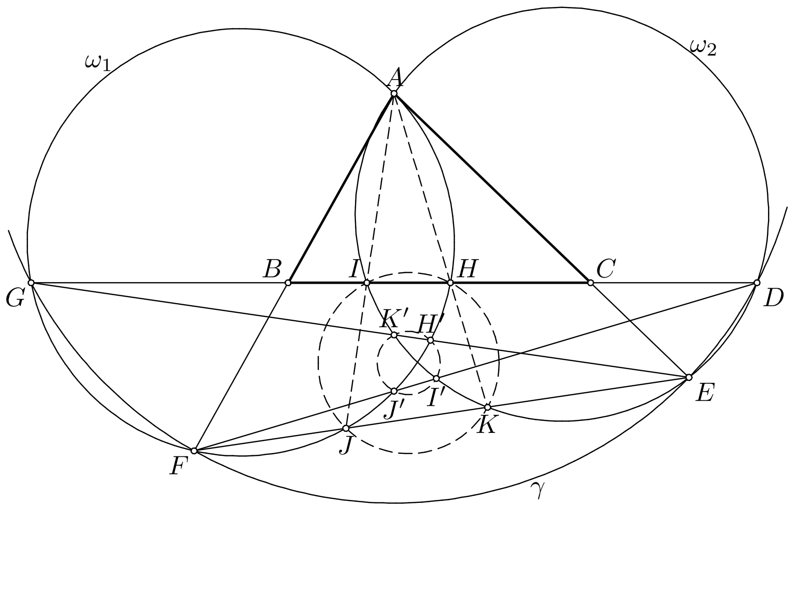
(Greece)

Solution.

From $\sphericalangle FAH = \sphericalangle FGH = \sphericalangle FGD = \frac{1}{2}\sphericalangle FAD = 90^\circ - \sphericalangle AFD$ we deduce that $AH \perp DF$. Similarly, $\sphericalangle DAI = 180^\circ - \sphericalangle DEI = 180^\circ - \sphericalangle DEF = \sphericalangle DGF = \frac{1}{2}\sphericalangle DAF$, so we also have $AI \perp DF$. Therefore, points A, H, I are collinear. Analogously, we find that the triples of points (A, K, J), (A, H', I') and (A, K', J') are collinear.

Quadrilateral $HIJK$ is cyclic because $\sphericalangle AIK = \sphericalangle ADK = \sphericalangle AGH = \sphericalangle AJH$. Analogously, quadrilateral $H'I'J'K'$ is cyclic.

Finally, since $\sphericalangle H'JH = \sphericalangle H'GH = \sphericalangle EGD = \sphericalangle EFD = \sphericalangle JFJ' = \sphericalangle JHJ'$, quadrilateral $HJJ'H'$ is an isosceles trapezoid with $HJ \parallel H'J'$, so the perpendicular bisectors of HJ and $H'J'$ coincide. Analogously, the perpendicular bisectors of IK and $I'K'$ coincide. Therefore the circumcenters of $HIJK$ and $H'I'J'K'$ coincide.



G6. In a triangle ABC with $AB = AC$, ω is the circumcircle and O its center. Let D be a point on the extension of BA beyond A. The circumcircle ω_1 of triangle OAD intersects the line AC and the circle ω again at points E and G, respectively. Point H is such that $DAEH$ is a parallelogram. Line EH meets circle ω_1 again at point J. The line through G perpendicular to GB meets ω_1 again at point N and the line through G perpendicular to GJ meets ω again at point L. Prove that the points L, N, H, G lie on a circle. *(Cyprus)*

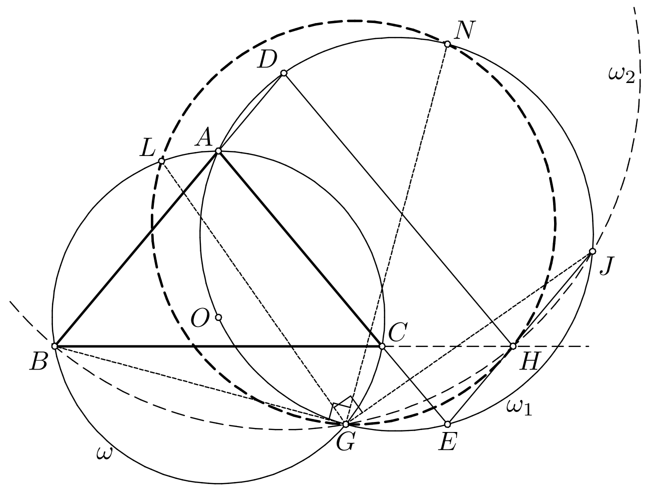
Solution.

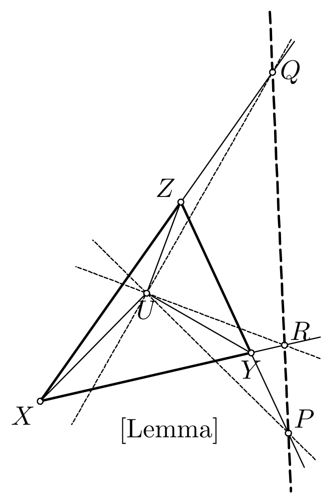
We first observe that $\sphericalangle DOE = \sphericalangle DAE = 2\sphericalangle ABC = \sphericalangle BOA$ and hence $\sphericalangle DOB = \sphericalangle EOA$, which together with $OB = OA$ and $\sphericalangle OBD = \sphericalangle BAO = \sphericalangle OAE$ gives us $\triangle OBD \cong \triangle OAE$. Therefore $BD = AE$.

Next, $OG = OA$ implies $\sphericalangle ODG = \sphericalangle ODA = \sphericalangle ODB$ and hence $\triangle OGD \cong \triangle OBD$. It follows that $DG = DB = AE = DH$. Moreover, since $AD \parallel EJ$, we have $DJ = AE = DG$. Thus, the points B, G, H, J lie on a circle ω_2 with center D.

We deduce that $\sphericalangle AGH = \sphericalangle BGH - \sphericalangle BGA = 180^\circ - \frac{1}{2}\sphericalangle HDB - \sphericalangle BCA = 180^\circ - \frac{1}{2}\sphericalangle CAB - \sphericalangle BCA = 90^\circ$.

We will now invert the diagram through G. By $\hat{X}$ we denote the image of any point X. The points $\hat{H}$, $\hat{L}$, $\hat{N}$ then lie on the lines $\hat{B}\hat{J}$, $\hat{A}\hat{B}$ and $\hat{A}\hat{J}$, respectively, such that $\sphericalangle \hat{A}G\hat{H} = \hat{B}G\hat{N} = \sphericalangle \hat{J}G\hat{L} = 90^\circ$. It remains to prove that $\hat{H}$, $\hat{L}$ and $\hat{N}$ are collinear, which follows from the following statement:





Lemma. Let XYZ be a triangle and let U be a point in the plane. If the lines through U perpendicular to UX, UY, UZ meet the lines YZ, ZX, XY respectively at points P, Q, R, then the points P, Q and R are collinear.

Proof. Here we assume that U is inside $\triangle XYZ$ and the angles XUY, YUZ and ZUX are all obtuse - the other cases are similar. We have

$$\frac{\overrightarrow{YP}}{\overrightarrow{PZ}} = -\frac{P_{YUP}}{P_{PUZ}}, \quad \frac{\overrightarrow{ZQ}}{\overrightarrow{QX}} = -\frac{P_{ZUQ}}{P_{QUX}}, \quad \frac{\overrightarrow{XR}}{\overrightarrow{RY}} = -\frac{P_{XUR}}{P_{RUY}}.$$

On the other hand, since $\sphericalangle QUX = \sphericalangle YUP$ are equal and equally directed, we have $\frac{P_{YUP}}{P_{QUX}} = \frac{UP \cdot UY}{UQ \cdot UX}$. Writing the analogous expressions for $\frac{P_{ZUQ}}{P_{RUY}}$ and $\frac{P_{XUR}}{P_{PUZ}}$

and multiplying them out we obtain $\frac{\overrightarrow{YP}}{\overrightarrow{PZ}} \cdot \frac{\overrightarrow{ZQ}}{\overrightarrow{QX}} \cdot \frac{\overrightarrow{XR}}{\overrightarrow{RY}} = -1$, and the result follows by Menelaus' theorem. □

Remark.

The result remains valid if D is any point on the line AB.

Point L does not depend on the choice of D. Indeed, $\sphericalangle LCB = \sphericalangle LGB = \sphericalangle JGB - 90^\circ = \sphericalangle JEA + \sphericalangle AGB - 90^\circ = \sphericalangle BAC + \sphericalangle ACB - 90^\circ = 90^\circ - \sphericalangle ABC$, so $CL \perp AB$.

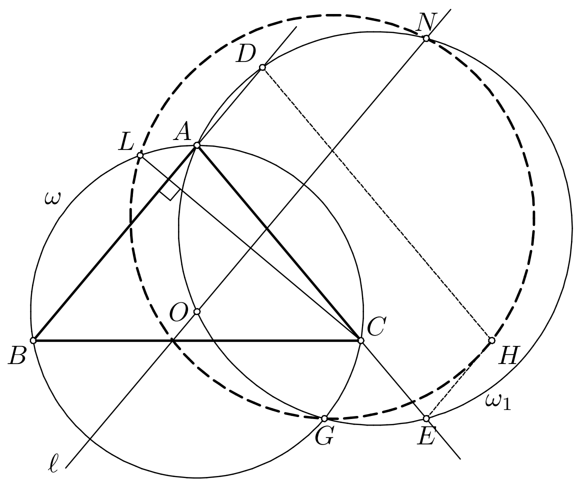
Also, since $\sphericalangle AON = \sphericalangle AGN = 90^\circ - \sphericalangle BGA = 90^\circ - \sphericalangle BCA = \sphericalangle OAB = \sphericalangle OND$, $ONDA$ is an isosceles trapezoid, i.e. $ON \parallel AB$.

Alternative formulation.

Based on the Remark, the PSC proposes the following modification which hides point J and defines the points in a more natural way:

A triangle ABC with $AB = AC$ is inscribed in a circle ω with center O. Its altitude from C meets ω again at point L. Line ℓ through O is parallel to AB. A circle ω_1 passes through points A and O and meets the lines AB, AC, ℓ and circle ω again at points D, E, N and G, respectively. Point H is such that $ADHE$ is a parallelogram.

Prove that H lies on the circumcircle of triangle GLN.



Number Theory

N1. For positive integers m and n, let $d(m,n)$ be the number of distinct primes that divide both m and n. For instance, $d(60,126)=d(2^2\times 3\times 5, 2\times 3^2\times 7)=2$. Does there exist a sequence (a_n) of positive integers such that:

(i) $a_1 \geqslant 2018^{2018}$;

(ii) $a_m \leqslant a_n$ whenever $m \leqslant n$;

(iii) $d(m,n)=d(a_m,a_n)$ for all positive integers $m \neq n$? *(United Kingdom)*

Solution.

Such a sequence does exist.

Let $p_1 < p_2 < p_3 < \ldots$ be the usual list of primes, and $q_1 < q_2 < \ldots$, $r_1 < r_2 < \ldots$ be disjoint sequences of primes greater than 2018^{2018}. For example, let $q_i \equiv 1$ and $r_i \equiv 3$ modulo 4. Then, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots$, where all but finitely many of the α_i will be zero, set

$$b_n := q_1^{\alpha_1} q_2^{\alpha_2} \cdots, \quad \text{for all } n \geqslant 2.$$

This sequence satisfies requirement (iii), but not the ordering conditions (i) and (ii). Iteratively, take $a_1 = r_1$, then given $a_1, \ldots, a_{n-1}$, define a_n by multiplying b_n by as large a power of r_n as necessary in order to ensure $a_n > a_{n-1}$. Thus $d(a_m,a_n)=d(b_m,b_n)=d(m,n)$, and so all three requirements are satisfied.

N2. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$n! + f(m)! \mid f(n)! + f(m!) \qquad (*)$$

for all $m, n \in \mathbb{N}$. *(Albania)*

Solution.

Answer: $f(n) = n$ for all $n \in \mathbb{N}$.

Taking $m = n = 1$ in $(*)$ yields $1 + f(1)! \mid f(1)! + f(1)$ and hence $1 + f(1)! \mid f(1) - 1$. Since $|f(1) - 1| < f(1)! + 1$, this implies $f(1) = 1$.

For $m = 1$ in $(*)$ we have $n! + 1 \mid f(n)! + 1$, which implies $n! \leqslant f(n)!$, i.e. $f(n) \geqslant n$.

On the other hand, taking $(m, n) = (1, p-1)$ for any prime number p and using Wilson's theorem we obtain $p \mid (p-1)! + 1 \mid f(p-1)! + 1$, implying $f(p-1) < p$. Therefore

$$f(p-1) = p - 1.$$

Next, fix a positive integer m. For any prime number p, setting $n = p - 1$ in $(*)$ yields $(p-1)! + f(m)! \mid (p-1)! + f(m!)$, and hence

$$(p-1)! + f(m)! \mid f(m!) - f(m)! \quad \text{for all prime numbers } p.$$

This implies $f(m!) = f(m)!$ for all $m \in \mathbb{N}$, so $(*)$ can be rewritten as $n! + f(m)! \mid f(n)! + f(m)!$. This implies

$$n! + f(m)! \mid f(n)! - n! \quad \text{for all } n, m \in \mathbb{N}.$$

Fixing $n \in \mathbb{N}$ and taking $m \in \mathbb{N}$ large enough, we conclude that $f(n)! = n!$, i.e. $f(n) = n$, for all $n \in \mathbb{N}$.

One readily checks that the identity function satisfies the conditions of the problem.

N3. Find all primes p and q such that $3p^{q-1}+1$ divides 11^p+17^p. *(Bulgaria)*

Solution.

Answer: $(p,q)=(3,3)$.

For $p=2$ it is directly checked that there are no solutions. Assume that $p>2$.

Observe that $N=11^p+17^p\equiv 4 \pmod 8$, so $8\nmid 3p^{q-1}+1>4$. Consider an odd prime divisor r of $3p^{q-1}+1$. Obviously, $r\notin\{3,11,17\}$. There exists b such that $17b\equiv 1 \pmod r$. Then $r\mid b^pN\equiv a^p+1 \pmod r$, where $a=11b$. Thus $r\mid a^{2p}-1$, but $r\nmid a^p-1$, which means that $\operatorname{ord}_r(a)\mid 2p$ and $\operatorname{ord}_r(a)\nmid p$, i.e. $\operatorname{ord}_r(a)\in\{2,2p\}$.

Note that if $\operatorname{ord}_r(a)=2$, then $r\mid a^2-1\equiv(11^2-17^2)b^2 \pmod r$, which gives $r=7$ as the only possibility. On the other hand, $\operatorname{ord}_r(a)=2p$ implies $2p\mid r-1$. Thus, all prime divisors of $3p^{q-1}+1$ other than 2 or 7 are congruent to 1 modulo $2p$, i.e.

$$3p^{q-1}+1=2^\alpha 7^\beta p_1^{\gamma_1}\cdots p_k^{\gamma_k}, \qquad (*)$$

where $p_i\notin\{2,7\}$ are prime divisors with $p_i\equiv 1 \pmod{2p}$.

We already know that $\alpha\leqslant 2$. Also, note that

$$\frac{11^p+17^p}{28}=11^{p-1}-11^{p-2}17+11^{p-3}17^2-\cdots+17^{p-1}\equiv p\cdot 4^{p-1} \pmod 7,$$

so 11^p+17^p is not divisible by 7^2 and hence $\beta\leqslant 1$.

If $q=2$, then $(*)$ becomes $3p+1=2^\alpha 7^\beta p_1^{\gamma_1}\cdots p_k^{\gamma_k}$, but $p_i\geqslant 2p+1$, which is only possible if $\gamma_i=0$ for all i, i.e. $3p+1=2^\alpha 7^\beta\in\{2,4,14,28\}$, which gives us no solutions.

Thus $q>2$, which implies $4\mid 3p^{q-1}+1$, i.e. $\alpha=2$. Now the right hand side of $(*)$ is congruent to 4 or 28 modulo p, which gives us $p=3$. Consequently $3^q+1\mid 6244$, which is only possible for $q=3$. The pair $(p,q)=(3,3)$ is indeed a solution.

N4. Let $P(x) = a_d x^d + \cdots + a_1 x + a_0$ be a non-constant polynomial with nonnegative integer coefficients having d rational roots. Prove that

$$\operatorname{lcm}\left(P(m), P(m+1), \ldots, P(n)\right) \geqslant m\binom{n}{m}$$

for all positive integers $n > m$. *(Iran)*

Solution.

Let $x_i = -\frac{p_i}{q_i}$ $(1 \leqslant i \leqslant d)$ be the roots of $P(x)$, where $p_i, q_i \in \mathbb{N}$ and $\gcd(p_i, q_i) = 1$. By Gauss' lemma, we have $P(x) = c(q_1 x + p_1)(q_2 x + p_2) \cdots (q_d x + p_d)$ for some $c \in \mathbb{N}$, so $q_1 x + p_1 \mid P(x)$. Thus it suffices to prove the statement for $P(x) = q_1 x + p_1 = qx + p$. Let

$$\begin{aligned} A &= \operatorname{lcm}(qm+p, q(m+1)+p, \ldots, qn+p) &&= \textstyle\prod_{i=1}^{s} p_i^{\alpha_i}, \\ B &= (qm+p)(q(m+1)+p)\cdots(qn+p) &&= \textstyle\prod_{i=1}^{s} p_i^{\beta_i} \end{aligned}$$

be the prime factorizations of A and B.

Consider a prime divisor p_i. We have $p_i^{\alpha_i} \mid qx + p$ for some $m \leqslant x \leqslant n$. On the other hand, if $p_i^r \mid qy + p$ $(r \leqslant \alpha_i)$ for some $m \leqslant y \leqslant n$ with $y \neq x$, then $p_i^r \mid q(x-y)$, i.e. $p_i^r \mid x - y$. Taking the product over all $y \neq x$ we obtain that

$$p_i^{\beta_i} \quad \text{divides} \quad p_i^{\alpha_i} \cdot \prod_{\substack{y=m \\ y \neq x}}^{n} |x-y|, \quad \text{which divides} \quad p_i^{\alpha_i}(n-m)!.$$

It follows that $B \mid A \cdot (n-m)!$, but $B \geqslant m(m+1)\cdots n$, so the result immediately follows.

N5. Let x and y be positive integers. If for each positive integer n we have that

$$(ny)^2 + 1 \mid x^{\varphi(n)} - 1,$$

prove that $x = 1$. *(Greece)*

Solution.

Let us take $n = 3^k$ and suppose that p is a prime divisor of $(3^k y)^2 + 1$ such that $p \equiv 2 \pmod 3$.

Since p divides $x^{\varphi(n)} - 1 = x^{2\cdot 3^{k-1}} - 1$, the order of x modulo p divides both $p-1$ and $2 \cdot 3^{k-1}$, but $\gcd(p-1, 2\cdot 3^{k-1}) \mid 2$, which implies that $p \mid x^2 - 1$. The result will follow if we prove that the prime p can take infinitely many values.

Suppose, to the contrary, that there are only finitely many primes p with $p \equiv 2 \pmod 3$ that divide a term of the sequence

$$a_k = 3^{2k} y^2 + 1 \quad (k \geqslant 0).$$

Let $p_1, p_2, \ldots, p_m$ be these primes. Clearly, we may assume without loss of generality that $3 \nmid y$. Then $a_0 = y^2 + 1 \equiv 2 \pmod 3$, so it has a prime divisor of the form $3s + 2$ $(s \in \mathbb{N}_0)$.

For $N = (y^2+1)p_1 \cdots p_m$ we have $a_{\varphi(N)} = 3^{2\varphi(N)} y^2 + 1 \equiv y^2 + 1 \pmod N$, which means that

$$a_{\varphi(N)} = (y^2+1)(tp_1 \cdots p_m + 1)$$

for some positive integer t. Since $y^2 + 1 \equiv 2 \pmod 3$ and $3^{2\varphi(N)} y^2 + 1 \equiv 1 \pmod 3$, the number $tp_1 \cdots p_m + 1$ must have a prime divisor of the form $3s+2$, but it cannot be any of the primes $p_1, \ldots, p_m$, so we have a contradiction as desired.

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Shortlisted Problems with Solutions

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