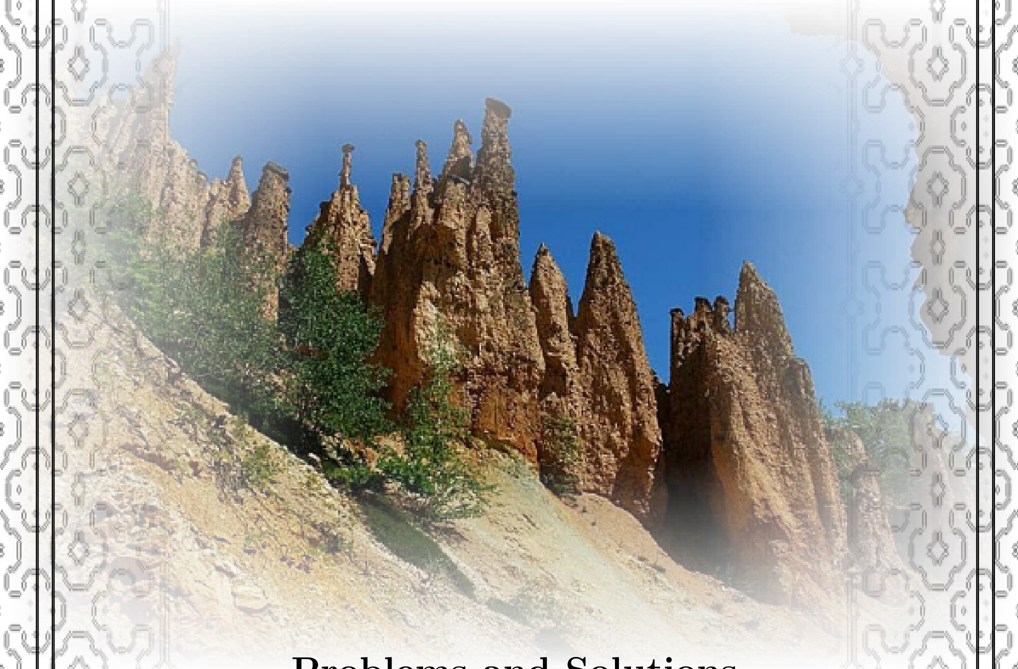
Serbian Mathematical Olympiad 2017

for high school students

Belgrade, March 31 – April 1



Problems and Solutions

Edited by Dušan Djukić

Cover photo: Djavolja Varoš

SHORT HISTORY AND SYSTEM

Mathematical society of Serbia runs mathematical competitions since 1958, including school, municipal, regional and republic rounds. The first federal level competition in the former Yugoslavia, which Serbia was a part of, was held in 1960. for senior high school students only - second and first grade students were added in 1970 and 1974, respectively. The team for international competitions used to be selected through the Federal competition, with an additional selection test when needed. After the breakdown of the old Yugoslavia, this system was continued in the newly formed FR Yugoslavia, later renamed Serbia and Montenegro, but with problems divided into A and B categories (specialized schools and others) since 1999. Since 2007, after the separation of Montenegro, the Federal competition has been replaced with a two-day Serbian Mathematical Olympiad, although ad-hoc team selection tests were still occasionally held. An additional two-day team selection competition has been officially added in 2017.

A mathematical competition season in Serbia now consists of the following rounds:

1. *Municipal round*, held in December or January, with 5 problems in 3 hours.
2. *Regional round*, held in February, again with 5 problems in 3 hours.
3. *State (republic) round*, held in March in a selected town in the country, with 4 or 5 problems (in A and B category) in 4 hours. There are 300 to 400 participants in total.
4. *Serbian Mathematical Olympiad* (SMO), held in late March/early April in the IMO format. The maximum score is 42. There are 30 to 35 participants. Top six are invited to the Balkan MO team.
5. *IMO Team Selection Competition*, held after the Balkan MO, again in the IMO format. The maximum score is 42. Top 12 from the SMO take part. The scores are added to those from SMO, where Balkan MO team participants get bonus points for gold (7) and silver medals (3) for the grand total of at most 91. Top six in the total score are invited to the IMO team.

The 11-th Serbian Mathematical Olympiad (SMO) for high school students took place in Belgrade on March 31 and April 1, 2017. There were 34 students from Serbia and 1 guest student from Republika Srpska (Bosnia and Herzegovina). The average score on the contest was 16.15 points. Problems 1, 2 and 4 were easier, and problems 3, 5 and 6 turned out to be difficult.

Top 12 students (and 1 unofficial) were invited to the IMO Team Selection Competition (TST) that took place in Belgrade on May 21-22. The average score on this test was 17.08 points. The team for the 57-th IMO was selected based on the results:

Name	School	Total score
Aleksa Milojević	Math High School, Belgrade	60 points
Pavle Martinović	Math High School, Belgrade	58 points
Igor Medvedev	Math High School, Belgrade	53 points
Jelena Ivančić	Math High School, Belgrade	51 points
Marko Medvedev	Math High School, Belgrade	47 points
Ognjen Tošić	Math High School, Belgrade	43 points

In this booklet we present the problems and full solutions of the Serbian Mathematical Olympiad, IMO Team Selection Competition and Balkan Mathematical Olympiad.

Serbian MO 2017 – Problem Selection Committee

- Vladimir Baltić
- Bojan Bašić *(chairman)*
- Dušan Djukić
- Miljan Knežević
- Nikola Petrović
- Marko Radovanović
- Miloš Stojaković

SERBIAN MATHEMATICAL OLYMPIAD

for high school students

Belgrade , 31.03.2017.

First Day

1. Let a, b and c be positive real numbers with $a+b+c=1$. Prove the inequality

$$a\sqrt{2b+1}+b\sqrt{2c+1}+c\sqrt{2a+1}\leqslant\sqrt{2-(a^2+b^2+c^2)}.$$

(Nikola Petrović)

2. A convex quadrilateral $ABCD$ is inscribed in a circle. The lines AD and BC meet at point E. Points M and N are taken on the sides AD i BC, respectively, so that $AM:MD=BN:NC$. Let the circumcircles of triangle EMN and quadrilateral $ABCD$ intersect at points X and Y. Prove that either the lines AB, CD and XY have a common point, or they are all parallel. *(Dušan Djukić)*

3. There are $2n-1$ bulbs in a line. Initially, the central (n-th) bulb is on, whereas all others are off. A step consists of choosing a string of at least three (consecutive) bulbs, the leftmost and rightmost ones being off and all between them being on, and changing the states of all bulbs in the string (for instance, the configuration $\bullet\circ\circ\circ\bullet$ will turn into $\circ\bullet\bullet\bullet\circ$). At most how many steps can be performed?

(Dušan Djukić)

Time allowed: 270 minutes.
Each problem is worth 7 points.

SERBIAN MATHEMATICAL OLYMPIAD

for high school students

Belgrade, 01.04.2017.

Second Day

4. Suppose that a positive integer a is such that, for any positive integer n, the number n^2a-1 has a divisor greater than 1 and congruent to 1 modulo n. Prove that a is a perfect square. *(Dušan Djukić)*

5. Determine the maximum number of queens that can be placed on a 2017×2017 chessboard so that each queen attacks at most one of the others.

(Bojan Bašić and PSC)

6. Let k be the circumcircle of triangle ABC, and let k_a be its excircle opposite to A. The two common tangents of k and k_a meet the line BC at points P and Q. Prove that $\sphericalangle PAB = \sphericalangle QAC$. *(Dušan Djukić)*

Time allowed: 270 minutes.
Each problem is worth 7 points.

SOLUTIONS

1. After squaring both sides and using the equality $1-a^2+b^2+c^2=2(ab+bc+ca)$, the required inequality becomes

$$\begin{aligned} L = \ & 2a^2b+2b^2c+2c^2a+ \\ & 2ab\sqrt{(2b+1)(2c+1)}+2bc\sqrt{(2c+1)(2a+1)}+2ca\sqrt{(2a+1)(2b+1)} \leqslant \\ R = \ & 4(ab+bc+ca). \end{aligned}$$

The AM-GM inequality gives us $2ab\sqrt{(2b+1)(2c+1)} \leqslant ab(2b+2c+2)$ and analogously $2bc\sqrt{(2c+1)(2a+1)} \leqslant bc(2c+2a+2)$ and $2ca\sqrt{(2a+1)(2b+1)} \leqslant ca(2a+2b+2)$, so summing these up yields

$$\begin{aligned} L \leqslant \ & 2(a^2b+b^2c+c^2a+ab^2+bc^2+ca^2+3abc)+2(ab+bc+ca) = \\ & 2(a+b+c+1)(ab+bc+ca)=4(ab+bc+ca)=R. \end{aligned}$$

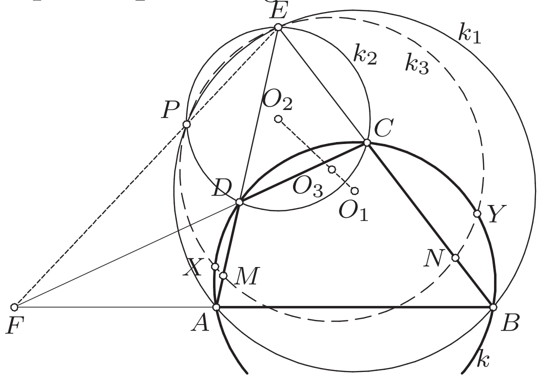
Second solution. Function $f(x)=\sqrt{x}$ is concave because $f'(x)=2/\sqrt{x}$ is decreasing. Then Jensen's inequality with weights a, b and c gives

$$\begin{aligned} a\sqrt{2b+1}+b\sqrt{2c+1}+c\sqrt{2a+1} & \leqslant \sqrt{a(2b+1)+b(2c+1)+c(2a+1)} \\ & = \sqrt{1+2(ab+bc+ca)} = \sqrt{2-(a^2+b^2+c^2)}. \end{aligned}$$

Remark. If we allow the numbers a,b,c to be zero, equality is attained if $a=b=c=\frac{1}{3}$ or $(a,b,c)=(1,0,0)$ with permutations.

2. If $AB \parallel CD$, the statement is trivial: points X and Y are symmetric with respect to the perpendicular bisector of AB and CD, and hence $AB \parallel XY \parallel CD$.

Assume $AB \not\parallel CD$. Then the circumcircles k_1 and k_2 of triangles EAB and ECD meet at a point $P \neq E$. Since $\sphericalangle PAD = \sphericalangle PBE$ and $\sphericalangle PDA = 180^\circ - \sphericalangle PDE = 180^\circ - \sphericalangle PCE = \sphericalangle PCB$, triangles PAD and PBC are similar. Moreover, point M in $\triangle PAD$ corresponds to point N in $\triangle PBC$, so $\sphericalangle PME = \sphericalangle PNE$. Therefore, points E, P, M and N lie on some circle k_3.



Since F has the same power $FA \cdot FB = FC \cdot FD$ to circles k_1, k_2 and the circumcircle k of $ABCD$, it lies on the radical axis EP of k_1 and k_2. Thus $FA \cdot FB = FE \cdot FP$, so F also lies on the radical axis of k_1 and k_3, which is line XY.

Second solution. Let k, k_1, k_2 and k_3 be the circumcircles of $ABCD$, EAB, ECD and EMN. We should prove that the radical centers of the triples (k, k_1, k_2) and (k, k_1, k_3) coincide (possibly at infinity). It suffices to show that circles k_1, k_2, k_3 are coaxial, i.e. that their centers O_1, O_2, O_3 are collinear.

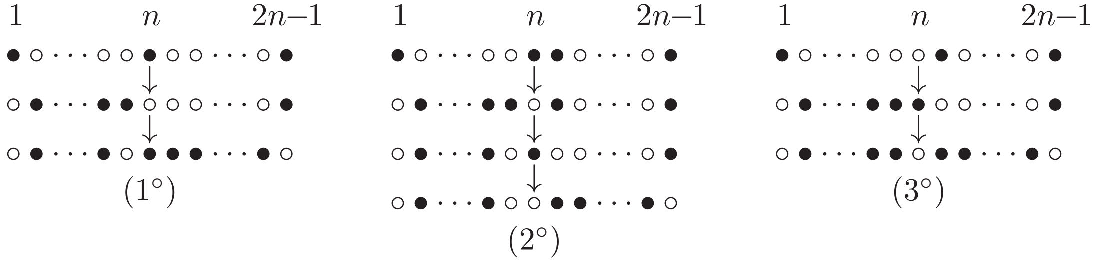
For $i = 1, 2, 3$, denote by E_i the point symmetric to E with respect to O_i. Let E_3' be the point on segment E_1E_2 such that $E_1E_3' : E_3'E_2 = AM : MD$. Since $E_1A \perp AD$ and $E_2D \perp AD$, Thales' theorem gives $E_3'M \perp AD$; analogously, we have $E_3'N \perp BC$, so $E_3' \equiv E_3$.

3. The answer is $\left[\frac{2^{n+1}-5}{3}\right]$.

Assign to the i-th bulb the number $2^{|i-n|}$ and define the *value* of a configuration as the sum of numbers assigned to the bulbs that are on. The initial configuration has value 1. With each step, the value increases by a multiple of 3. If a step switches the n-th bulb, the value increases by exactly 3; we call such steps *good*.

Since the value cannot exceed $2^{n+1} - 4$ (for not all bulbs can be on), one cannot make more than $[\frac{2^{n+1}-5}{3}]$ steps. In order to show that this number can be attained, it suffices to show that at least $\frac{2^{n+1}-7}{3}$ good steps can be made.

We prove by induction on n that, starting with a configuration of value at most 3, we can reach a configuration of value at least $2^{n+1} - 6$ by a sequence of good steps. For $n \leqslant 2$ this is directly verified. Let $n \geqslant 3$. By the inductive hypothesis for $n-1$, it is possible to reach a configuration with the first and last bulb off, and value at least $2^n - 6$. In such a configuration, other than the outer two, the bulbs that are off can be (1°) only the n-th, (2°) only the n-th and an adjacent one, or (3°) only one bulb adjacent to the n-th. In each of these three cases, in at most three good steps we reach a configuration in which the two outer bulbs are on and the value of the rest of the configuration (not counting these two) is at most 3.



Now we can apply the inductive hypothesis for $n-1$ again, finishing the induction.

4. Let $n^2a-1=(nx_n+1)d_n$ $(x_n,d_n\in\mathbb{N})$. Then $d_n\equiv -1 \pmod{n}$, which means that
$$n^2a-1=(nx_n+1)(ny_n-1)\quad\text{for some } x_n,y_n\in\mathbb{N},$$
which is equivalent to $na-nx_ny_n=y_n-x_n>-x_ny_n$. It follows that $x_n\leqslant x_ny_n<\frac{n}{n-1}a\leqslant 2a$. Therefore, there is a number X that occurs infinitely often in the sequence $x_1,x_2,\dots$. Thus for infinitely many values of n we have $nX+1\mid n^2a-1$ and hence
$$nX+1\mid X^2(n^2a-1)-a(n^2x^2-1)=a-X^2.$$
This is only possible if $a-X^2=0$, i.e. $X^2=a$.

Second solution. As in the first solution, let $n^2a-1=(nx_n+1)(ny_n-1)$, i.e. $y_n-x_n=n(a-x_ny_n)=nd_n$. We distinguish three cases.

(1°) If $d_n>0$, then $a=d_n+x_n(x_n+nd_n)>nd_nx_n$, which is impossible for $n\geqslant a$.

(2°) If $d_n<0$, then $a=d_n+y_n(y_n-nd_n)=y_n^2-d_n(ny_n-1)>ny_n-1$, which is impossible for $n\geqslant a+1$.

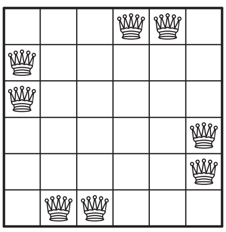
(3°) If $d_n=0$, then $a=x_n^2$ is a perfect square

5. Denote $n=2017$. Suppose that we have placed $m>n$ queens. No row contains more than two queens, so there are at least $m-n$ rows with two queens, and hence at most $m-2(m-n)=2n-m$ are alone in their rows. Similarly, at most $2n-m$ queens are alone in their columns. On the other hand, every queen is alone in its row or column, which means that $m\leqslant 2(2n-m)$. Thus $m\leqslant [\frac{4n}{3}]=2689$.

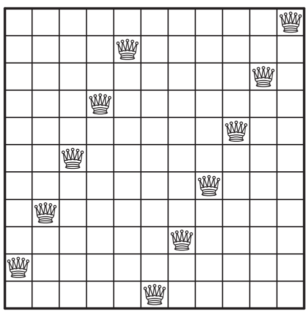
Image A shows 8 queens placed on a 6×6 board under the problem condition. Before proceeding to the 2017×2017 board, consider the following arrangement:

- On a 335×335 board, one can place 335 queens so that they do not attack each other *even if the diagonals are extended modulo* 335. Indeed, it suffices to place queens on the squares (x,y) with $1\leqslant x,y\leqslant 335$ and $y\equiv 2x \pmod{335}$, as on image B. Then all the sums $x+y$ are different modulo 335 and so are the differences $x-y$, so no two queens are in the same row, column or extended diagonal.

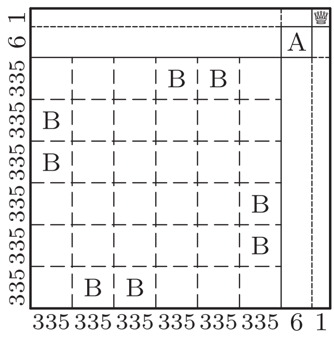
A - 6×6 board



B - 335×335 board



C - 2017×2017 board



Let us divide the 2017×2017 board into rectangles and squares with sides 335, 6 and 1, as on image C. We arrange queens on the squares denoted B and A as shown on images B and A, and put one more queen in the upper-right cell. Thus we have placed $8 \cdot 335 + 8 + 1 = 2689$ queens in total. It is easily verified that this arrangement fulfills the requirements.

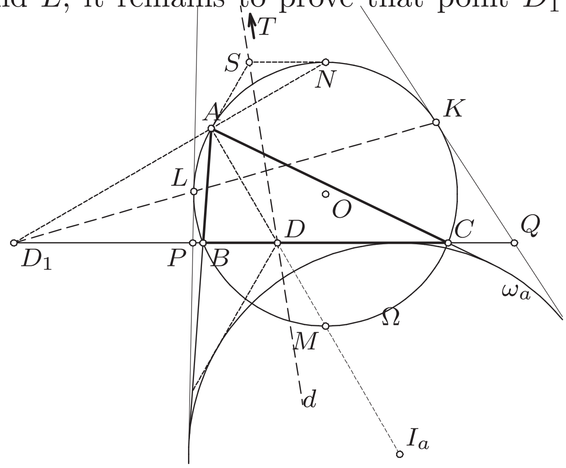
Remark. The $n \times n$ board whose diagonals are extended modulo n is known as a *torus board*. It is known that it is possible to place n mutually non-attacking queens on a torus board $n \times n$ if and only if $n \equiv \pm 1 \pmod 6$.

6. Let the interior and exterior bisector of angle BAC respectively meet the line BC at points D and (possibly infinite) D_1. The common tangents meet at the center T of positive homothety $\mathcal{H}$ which maps the excircle ω_a to the circumcircle Ω. If T is an infinite point, then $\mathcal{H}$ is a translation and the proof remains unchanged.

Lemma. Let an arbitrary line p through D_1 intersect Ω at points K and L. The tangents to Ω at K and L meet line BC at points P i Q, respectively. Then $\sphericalangle PAB = \sphericalangle CAQ$.

Proof. If D_1 is an infinite point, the statement is trivial by symmetry. Otherwise, from $\triangle PBK \sim \triangle PKC$ we get $\frac{PB}{PK} = \frac{PK}{PC} = \frac{KB}{KC}$ and hence $\frac{PB}{PC} = \left(\frac{KB}{KC}\right)^2$. Similarly, $\frac{QB}{QC} = \left(\frac{LB}{LC}\right)^2$. Since $\frac{KB}{KC} \cdot \frac{LB}{LC} = \frac{[KLB]}{[KLC]} = \frac{D_1B}{D_1C} = \frac{AB}{AC}$, it follows that $\frac{PB}{PC} \cdot \frac{QB}{QC} = \left(\frac{AB}{AC}\right)^2$, which is equivalent to $\sphericalangle PAB = \sphericalangle CAQ$. □

If the common tangents touch Ω at K and L, it remains to prove that point D_1 lies on the line KL, which is the polar of point T in Ω. By the duality principle, it is enough to prove that T lies on the polar d of point D_1 in Ω.



Denote by N the midpoint of arc BAC of circle Ω. Homothety $\mathcal{H}$ maps point D to the intersection point S of tangents to Ω at A and N, so the line DS passes through T. On the other hand, point D lies on the polar d because the quadruple $(B, C; D_1, D)$ is harmonic, whereas point S lies on d because the polar of S in Ω, which is line AN, passes through D_1. Hence the lines DS and d coincide, completing the proof.

Second solution. Let the common tangents touch Ω at K and L, where LP is the tangent closer to B. Denote by M the midpoint of arc BC not containing A, and by O and I_a the circumcenter and excenter opposite to A, respectively.

Since $\sphericalangle LPI_a = 90^\circ + \frac{1}{2}\sphericalangle LPC$ and $\sphericalangle LAI_a = \sphericalangle LAM = \frac{1}{2}\sphericalangle LOM = \frac{1}{2}\sphericalangle LPD_1 = 90^\circ - \frac{1}{2}\sphericalangle LPC$, it follows that $\sphericalangle LPI_a + \sphericalangle LAI_a = 180^\circ$, i.e. the quadrilateral $ALPI_a$ is cyclic. Similarly, $AKQI_a$ is also cyclic. Now $\sphericalangle PAI_a = \sphericalangle PLI_a = \sphericalangle QKI_a = \sphericalangle QAI_a$ because the angles PLI_a and QKI_a are symmetric in OI_a, and hence $\sphericalangle PAB = \sphericalangle QAC$.

IMO TEAM SELECTION COMPETITION

Belgrade, 21.05.2017.

First Day

1. Let D be the midpoint of side BC of a triangle ABC. Points E and F are taken on the respective sides AC and AB such that $DE = DF$ and $\sphericalangle EDF = \sphericalangle BAC$. Prove that

$$DE \geqslant \frac{AB + AC}{4}\,.$$

(Dušan Djukić)

2. Given an ordered pair of positive integers (x, y) with exactly one even coordinate, a *step* maps this pair to $(\frac{x}{2}, y + \frac{x}{2})$ if $2 \mid x$, and to $(x + \frac{y}{2}, \frac{y}{2})$ if $2 \mid y$. Prove that, for every odd positive integer $n > 1$ there exists an even positive integer b, $b < n$, such that after finitely many steps the pair (n, b) maps to the pair (b, n).

(Bojan Bašić)

3. Call a function $f : \mathbb{N} \to \mathbb{N}$ *lively* if

$$f(a + b - 1) = \underbrace{f(f(\cdots f}_{a \text{ times}}(b)\cdots))\text{for all } a, b \in \mathbb{N}.$$

Suppose that g is a lively function such that $g(A+2018) = g(A)+1$ holds for some $A \geqslant 2$.

(a) Prove that $g(n + 2017^{2017}) = g(n)$ for all $n \geqslant A + 2$.

(b) If $g(A + 2017^{2017}) \neq g(A)$, determine $g(n)$ for $n \leqslant A - 1$.

(Marko Radovanović)

Time allowed: 270 minutes.
Each problem is worth 7 points.

IMO TEAM SELECTION COMPETITION

Belgrade, 22.05.2017.

Second Day

4. An $n \times n$ square is divided into unit squares. One needs to place a number of isosceles right triangles with hypotenuse 2, with vertices at grid points, in such a way that every side of every unit square belongs to exactly one triangle (i.e. lies inside it or on its boundary). Determine all numbers n for which this is possible.
(Dušan Djukić)

5. For a positive integer $n \geqslant 2$, let $C(n)$ be the smallest positive real constant such that there is a sequence of n real numbers $x_1, x_2, \ldots, x_n$, not all zero, satisfying the following conditions:

(i) $x_1 + x_2 + \cdots + x_n = 0$;

(ii) for each $i = 1, 2, \ldots, n$, it holds that $x_i \leqslant x_{i+1}$ or $x_i \leqslant x_{i+1} + C(n)x_{i+2}$ (the indices are taken modulo n).

Prove that:

(a) $C(n) \geqslant 2$ for all n;

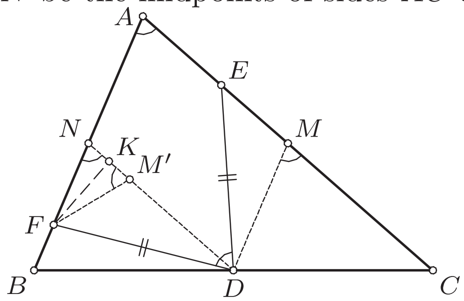
(b) $C(n) = 2$ if and only if n is even. *(Dušan Djukić)*

6. Let k be a positive integer and let n be the smallest positive integer having exactly k divisors. If n is a perfect cube, can the number k have a prime divisor of the form $3j + 2$? *(Bojan Bašić)*

Time allowed: 270 minutes.
Each problem is worth 7 points.

SOLUTIONS

1. We may assume that $AB \leqslant AC$. Let M and N be the midpoints of sides AC and AB, respectively, and let M' be the point on segment DN such that $DM' = DN$. Since $\sphericalangle M'DF = \sphericalangle MDE$, triangles DME and $DM'F$ are congruent, so $\sphericalangle FM'N = 180° - \sphericalangle DM'F = 180° - \sphericalangle DME = \sphericalangle BAC = \sphericalangle FNM'$, which means that $\triangle FM'N$ is isosceles. The midpoint K of $M'N$ is also the foot of the perpendicular from F to $M'N$, so $DF \geqslant DK = \frac{DM'+DN}{2} = \frac{AB+AC}{4}$.



Second solution. Assume that E lies on segment AM. Denote $x = \sphericalangle MDE = \sphericalangle NDF$. By the law of sines in $\triangle MDE$ and $\triangle NDF$ we have $\frac{DM}{DE} = \frac{\sin(\alpha-x)}{\sin\alpha}$ and $\frac{DN}{DF} = \frac{\sin(\alpha+x)}{\sin\alpha}$, and therefore $\frac{b+c}{4DE} = \frac{DM+DN}{2DE} = \frac{\sin(\alpha+x)+\sin(\alpha-x)}{2\sin\alpha} = \cos x \leqslant 1$.

Remark. One obtains $4 \cdot DE = \sqrt{(b+c)^2 + (b-c)^2 \operatorname{tg}^2\alpha}$.

2. Denote by (x_k, y_k) the pair obtained after k steps. The sum $x_k + y_k$ is invariant and equals $s = n+b$. Since $2\cdot(x+\frac{y}{2}) \equiv 2\cdot\frac{x}{2} \equiv x \pmod{x+y}$, we have $2x_k \equiv x_{k-1} \pmod{s}$. A simple induction yields

$$2^k x_k \equiv x_0 = n \pmod{s}.$$

Since $(s, 2^k) = 1$, it is enough to prove the existence of an odd number s, $n < s < 2n$, such that for some k we have $2^k b \equiv n \pmod{s}$, i.e. $(2^k+1)n \equiv 0 \pmod{s}$. To this end, one can simply take $s = 2^r + 1$ and $k = r$, where $2^{r-1} < n < 2^r$ ($r \in \mathbb{N}$). Thus $b = 2^r + 1 - n$.

Remark. Clearly, one can take any s such that $s \mid 2^k + 1$ for some $k \in \mathbb{N}$. For example, $s = 3^i 11^j$ ($i, j \in \mathbb{N}_0$) works. From here, one can deduce that, given any constants $0 < \alpha < \beta$, for all big enough n, there is a desired number b with $\alpha n < b < \beta n$.

3. If $g(a) = g(a+d)$ for some $a, d \in \mathbb{N}$, the problem condition gives $g(a+n) = g^{n+1}(a) = g^{n+1}(a+d) = g(a+n+d)$, implying that function $g(x)$ is periodic with period d for $x \geqslant a$. Such a and d actually exist: Indeed, $g(A+2019) =$

$g(g(A+2018)) = g(g(A)+1) = g(g(g(A))) = g(A+2)$, so it follows from above that $g(n+2017) = g(n)$ for $n \geqslant A+2$. Consider the smallest d for which such an a exists; let $a = a_0$ be the smallest such a. Since d is minimal, for $x, y \geqslant a_0$ it holds that $g(x) = g(y)$ if and only if $d \mid x - y$. Clearly, $d \mid 2017$.

We immediately have $g(n + 2017^{2017}) = g(n)$ for $n \geqslant A + 2$. On the other hand, since $g(A + 2017^{2017}) \neq g(A)$, we deduce that $A \leqslant a_0 - 1$, i.e. $a_0 \in \{A+1, A+2\}$.

Suppose that $g(a') = g(a' + d')$ for some $a' \leqslant a_0 - 1$ and some $d' \in \mathbb{N}$. Then function $g(x)$ has period d' for $x \geqslant a'$, which implies $d \mid d'$, but then we have $g(a_0 - 1) = g(a_0 - 1 + d') = g(a_0 - 1 + d)$, contradicting the minimality of a_0.

Therefore, if $g(x) = g(y)$ and $x \leqslant a_0 - 1$, then $x = y$. Now the equality $g(g(n)) = g(n+1)$ for $n + 1 \leqslant A \leqslant a_0 - 1$ implies $g(n) = n + 1$.

Remark. The problem condition is equivalent to $g(g(n)) = g(n+1)$ for all n.

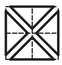
For $n \geqslant a_0 - 1$, equation $g(g(n)) = g(n+1)$ implies $g(n) \equiv n + 1 \pmod{d}$. Thus every lively function g is of the form

$$g(n) = \begin{cases} n+1 & \text{for } n \leqslant a_0 - 2; \\ a_0 + \alpha d & \text{for } n = a_0 - 1; \\ a_0 + i + \beta_i d & \text{for } n \geqslant a_0 \text{ and } n \equiv a_0 + i \pmod{d},\ 0 \leqslant i < d - 1, \end{cases}$$

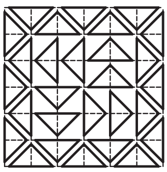
where $\alpha, \beta_0, \dots, \beta_{d-1}$ are arbitrary positive integers. It is straightforward to verify that such a function is lively.

4. There are $2n(n+1)$ unit segments on the grid, and each triangle covers three, so we must have $3 \mid 2n(n+1)$, i.e. $n \equiv 0$ or $n \equiv 2 \pmod{3}$. Moreover, the segments on the boundary of the big square have to be covered by hypotenuses, so we must also have $2 \mid n$. Therefore, $n \equiv 0$ or $n \equiv 2 \pmod{6}$.

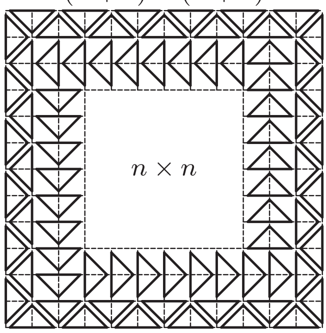
2×2



6×6



$(n+6) \times (n+6)$



The left image shows desired arrangements of triangles for $n = 2$ and $n = 6$, whereas the right image shows that every arrangement of triangles on an $n \times n$ squares can be extended to $(n+6) \times (n+6)$. A simple induction shows that a desired arrangement exists whenever $n = 6k$ or $n = 6k - 4$ for some $k \in \mathbb{N}$.

5. The sequence cannot contain two adjacent non-positive terms. Indeed, if $a_{i-1} > 0 \geqslant a_i, a_{i+1}$, then $a_{i-1} > \max\{a_i, a_i + C(n)a_{i+1}\}$, contradicting (ii). Hence, the sequence consists of blocks of positive terms followed by a single non-positive term.

Consider an arbitrary block of positive terms $a_k, a_{k+1}, \dots, a_{k+l-1}$. We call the term a_k *initial* and a_{k+l-1} *final*. Let P be the sum of all initial terms in the sequence, K be the sum of the final ones, N be the sum of all non-positive terms, and S be the sum of positive terms which are not initial or final.

Summing the inequalities $a_{k+l-1} \leqslant a_{k+l} + C(n)a_{k+l+1}$ over all blocks yields $K \leqslant N + C(n)P$. Since $N = -K - S - P$, this relation becomes

$$2K \leqslant (C(n)-1)P - S. \qquad (*)$$

Suppose now that $C(n) \leqslant 2$. Summing the inequalities $a_{k+2i} \leqslant a_{k+2i+1} + 2a_{k+2i+2}$ for $0 \leqslant i \leqslant [\frac{l-3}{2}]$ and adding the inequality $a_{k+l-2} \leqslant 2a_{k+l-1}$ if $2 \mid l$, we obtain $a_k \leqslant a_{k+1} + a_{k+2} + \dots + a_{k+l-2} + 2a_{k+l-1}$. Summing over all blocks then yields

$$P \leqslant S + 2K. \qquad (**)$$

Equality in $(**)$ is possible only if the length l of the block is odd. Indeed, if $2 \mid l$, all inequalities participating in the sum must be equalities, so in particular $a_{k+l-2} = 2a_{k+l-1}$, which contradicts the condition $a_{k+l-2} \leqslant a_{k+l-1} > 0$.

Summing $(*)$ and $(**)$ gives us $0 \leqslant (C(n)-2)P$, and therefore $C(n) \geqslant 2$. Moreover, if $C(n) = 2$, all blocks must have odd lengths, which implies that n is even. Conversely, the example $x_r = (-1)^r$ shows that $C(n) = 2$ for even n.

Remark. In the case $2 \mid n$ there are nontrivial examples: For instance, if $n = 4$, one can take $(x_1, x_2, x_3, x_4) = (3a+2b, a, a+b, -5a-3b)$ for $a \in \mathbb{R}$ and $b \geqslant 0$.

It can be shown that $C(3) = 3$, $C(5) = \frac{1+\sqrt{11}}{2}$ and $\lim_{n\to\infty} C(n) = 2$.

6. Suppose that such a k exists. Let $p_1 < p_2 < \dots$ be all primes in the increasing order and let $n = \prod_{i=1}^m p_i^{\alpha_i}$ ($\alpha_m > 0$), where $k = (\alpha_1+1)\cdots(\alpha_m+1)$ and $3 \mid \alpha_i$ for all i. By the minimality of n we have $\alpha_1 \geqslant \dots \geqslant \alpha_m > 0$.

Lemma. Suppose that $\alpha_r + 1 = ab$ for $a, b \in \mathbb{N} \setminus \{1\}$. If $p_s < p_r^a < p_{s+1}$, then $\alpha_s \geqslant b-1 \geqslant \alpha_{s+1}$.

Proof. The number $n_1 = p_r^{(\alpha_s+1)a-1} p_s^{b-1} \prod_{i \notin \{r,s\}} p_i^{r_i}$ also has k divisors, so it satisfies $n_1 \geqslant n$. However, this reduces to $(p_r^a/p_s)^{\alpha_s-b+1} \geqslant 1$, i.e. $\alpha_s \geqslant b-1$.

Similarly, $n_1' = p_r^{(\alpha_{s+1}+1)a-1} p_{s+1}^{b-1} \prod_{i \notin \{r,s+1\}} p_i^{r_i} \geqslant n$ yields $\alpha_{s+1} \leqslant b-1$. □

Consider the largest r such that $\alpha_r + 1 = ab$ for some $a \equiv b \equiv 2 \pmod 3$, and let s and t be such that $p_s < p_r^a < p_{s+1}$ and $p_t < p_r^b < p_{t+1}$. Observe that Bertrand's postulate implies $\frac{1}{2}p_r^a < p_s$ and $p_{s+1} < 2p_r^a$, hence

$$p_r^{a-1} < p_s < p_r^a < p_{s+1} < p_r^{a+1} \quad \text{and analogously} \quad p_r^{b-1} < p_t < p_r^b < p_{t+1} < p_r^{b+1}.$$

It follows that $s, t > r$. We similarly obtain $|s - t| \neq 1$.

By the Lemma (since $3 \mid \alpha_i$) we have $\alpha_s > b - 1 > \alpha_{s+1}$ and $\alpha_t > a - 1 > \alpha_{t+1}$. Number $n_2 = p_r^{(\alpha_s+1)(\alpha_{t+1}+1)-1} p_s^{b-1} p_{t+1}^{a-1} \prod_{i \notin \{r,s,t+1\}} p_i^{r_i}$ also has k divisors, so $n_2 \geqslant n$, i.e.

$$1 \leqslant \frac{n_2}{n} = \frac{p_r^{(\alpha_s+1)(\alpha_{t+1}+1)-ab} p_{t+1}^{a-1-\alpha_{t+1}}}{p_s^{\alpha_s - b+1}} < \frac{p_r^{(\alpha_s+1)(\alpha_{t+1}+1)-ab+(b+1)(a-1-\alpha_{t+1})}}{p_r^{(a-1)(\alpha_s-b+1)}} = p_r^{1-(\alpha_s-b)(a-2-\alpha_{t+1})},$$

whence $(\alpha_s - b)(a - 2 - \alpha_{t+1}) < 1$. Since $3 \mid \alpha_s \neq b$, we must have $\alpha_{t+1} = a - 2$. By the assumption, $\alpha_i + 1$ has no divisors of the form $3j + 2$ if $i > r$, so if must odd. In particular, $2 \mid \alpha_{t+1}$, so $2 \mid a$. Analogously, $2 \mid b$, so $\alpha_r = 4c - 1$ for some $c \in \mathbb{N}$.

Since $2 \mid \alpha_m$ implies $\alpha_m > 3 > 1 > \alpha_{m+1} = 0$, the Lemma for $(a, b) = (2, 2c)$ and $(a, b) = (4, c)$ respectively gives us $p_m < p_r^{2c} < p_{m+1}$ and $p_m < p_r^c < p_{m+1}$. However, this is impossible, as by the Bertrand's postulate the interval (p_r^c, p_r^{2c}) contains at least one prime.

The 34-th Balkan Mathematical Olympiad was held from May 2 to May 7 in Ohrid in FYR Macedonia. The results of the Serbian contestants are shown below:

	1	2	3	4	Total	
Nikola Pavlović	10	10	10	0	30	Bronze medal
Marko Medvedev	10	10	10	4	34	Silver medal
Igor Medvedev	10	10	10	9	39	Gold medal
Aleksa Milojević	10	10	10	10	40	Gold medal
Jelena Ivančić	10	10	10	0	30	Bronze medal
Pavle Martinović	10	10	10	5	35	Silver medal

It turned out that all official contestants with at least 30 points scored the maximal 30 on the first three problems. The fourth problem was the one making difference. Thus the high medal cut-offs were expected: 7 contestants (6 official + 1 guest) with 39-40 points were awarded gold medals, 21 (16+5) with 31-38 points were awarded silver medals, and 44 (21+23) with 16-30 points were awarded bronze medals.

Here is the (unofficial) team ranking:

Member Countries		Guest Teams	
1. Bulgaria	226	Italy	187
2. Serbia	208	Kazakhstan	179
3. Romania	182	Turkmenistan	132
4. Greece	174	Saudi Arabia	128
5. Bosnia and Herzegovina	169	Azerbaijan	114
6. Turkey	128	United Kingdom	98
7. Moldova	127	Macedonia B	36
8. Macedonia	104	Kyrgyzstan	34
9. Montenegro	87	Qatar	11
10. Albania	60		
11. Cyprus	46		

BALKAN MATHEMATICAL OLYMPIAD

Ohrid, FYR Macedonia, 04.05.2017.

1. Find all ordered pairs (x, y) of positive integers such that:

$$x^3 + y^3 = x^2 + 42xy + y^2$$

(Moldova)

2. Let ABC be a triangle with $AB < AC$, and let Γ be its circumcircle. The tangent to Γ at C and the line through B parallel to AC intersect at D. The tangent to Γ at B and the line through C parallel to AB intersect at E. The tangents to Γ at B and C intersect at L. The circumcircle of triangle BDC meets AC again at T and the circumcircle of triangle BEC meets AB again at S.

Prove that the lines ST, BC and AL are concurrent. *(Greece)*

3. Let $\mathbb{N}$ be the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that:

$$n + f(m) \text{ divides } f(n) + nf(m)$$

for all $m, n \in \mathbb{N}$. *(Albania)*

4. There are $n > 2$ students sitting at a round table. Initially each student has exactly one candy. At each step, each student chooses one of the following operations:

(i) Pass one candy to the student on their left or the student on their right.

(ii) Divide all their candies into two, possibly empty, sets and pass one set to the student on their left and the other to the student on their right.

At each step the students perform their chosen operations simultaneously. An arrangement of candies is *legal* if it can be obtained in a finite number of steps. Find the number of legal arrangements.

(Two arrangements are different if there is a student who has different numbers of candies in each one.) *(Cyprus)*

Time allowed: 270 minutes.
Each problem is worth 10 points.

SOLUTIONS

1. Let $x = dx_1$ and $y = dy_1$, where $d = (x, y)$. The given equation rewrites as

$$d(x_1 + y_1)(x_1^2 - x_1y_1 + y_1^2) = x_1^2 - x_1y_1 + y_1^2 + 43x_1y_1, \qquad (*)$$

which implies that $x_1^2 - x_1y_1 + y_1^2 \mid 43x_1y_1$. Since $(x_1, x_1^2 - x_1y_1 + y_1^2) = (y_1, x_1^2 - x_1y_1 + y_1^2) = (x_1, y_1) = 1$, it follows that $x_1^2 - x_1y_1 + y_1^2 \mid 43$. Note that $x_1^2 - x_1y_1 + y_1^2 > 0$.

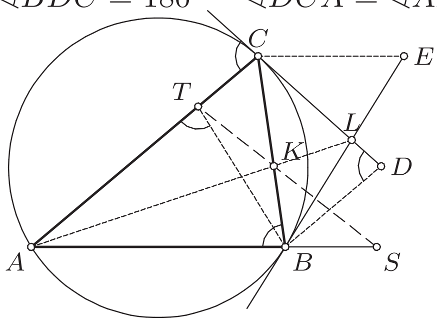
If $x_1^2 - x_1y_1 + y_1^2 = 1$, we must have $x_1 = y_1 = 1$, so $(*)$ yields $x = y = d = 22$.

The other possibility is $x_1^2 - x_1y_1 + y_1^2 = 43$, i.e. $(2x_1 - y_1)^2 + 3y_1^2 = 172$. Assuming w.l.o.g. that $x_1 \leqslant y_1$, we get $3y_1^2 \leqslant 172 \leqslant 4y_1^2$, which holds only for $y_1 = 7$. Then $x_1 = 1$ and $d = 1$, or $x_1 = 6$ and $d = \frac{43}{13}$. Only the first possibility gives a solution: $(x, y) = (1, 7)$.

Thus, the solutions (x, y) are pairs $(1, 7)$, $(7, 1)$ and $(22, 22)$.

2. Denote $AC = b$ and $AB = c$. Since $\sphericalangle BTA = \sphericalangle BDC = 180° - \sphericalangle DCA = \sphericalangle ABC$, triangles ATB and ABC are similar, and hence $\frac{AT}{AB} = \frac{AB}{AC}$, i.e. $AT = \frac{c^2}{b}$ and $TC = b - AT = \frac{b^2-c^2}{b}$. We similarly get $AS = \frac{b^2}{c}$ and $SB = \frac{b^2-c^2}{c}$.

On the other hand, line AL is a symmedian in triangle ABC, so it meets side BC at point K with $\frac{BK}{KC} = \frac{c^2}{b^2}$. Therefore $\frac{\overrightarrow{AS}}{\overrightarrow{SB}} \cdot \frac{\overrightarrow{BK}}{\overrightarrow{KC}} \cdot \frac{\overrightarrow{CT}}{\overrightarrow{TA}} = \frac{b^2}{c^2-b^2} \cdot \frac{c^2}{b^2} \cdot \frac{b^2-c^2}{c^2} = -1$ and by Menelaus' theorem point K lies on line ST.



3. The problem condition gives us $n + f(m) \mid f(n) + nf(m) - n(n + f(m)) = f(n) - n^2$. Setting $m = n = 1$ yields $f(1) + 1 \mid f(1) - 1$ and hence $f(1) = 1$.

Function $f(x) \equiv x^2$ satisfies the conditions. Suppose that $f(n_0) \neq n_0^2$ for some n_0. Then from $n_0 + f(m) \mid |f(n_0) - n_0^2|$ it follows that $f(m) \leqslant A = |f(n_0) - n_0^2| - n_0$. On the other hand, the problem condition for $m = 1$ yields $n + 1 \mid f(n) + n$, i.e. $f(n) \equiv 1 \pmod{n}$, but $f(n) < A + 1$, so we must have $f(n) = 1$ for all $n \geqslant A$.

Finally, for each $n > A$ it holds that $n + f(m) \mid f(m)(n + f(m)) - (f(n) + nf(m)) = f(m)^2 - 1$, so $f(m) = 1$ for all $m \in \mathbb{N}$, which is also a solution.

Therefore, the only solutions are functions $f(x) \equiv x^2$ and $f(x) \equiv 1$.

4. We start by noting that the number of all arrangements is $\binom{2n-1}{n}$.

Lemma. In two steps, any student can pass a candy (assuming he has one) to the student sitting two places to the left or right, so that the numbers of candies at other students do not change.

Proof. Consider three adjacent students A, B and C left-to-right, with A having at least one candy. Let all students perform operations of type (ii). Then each candy arbitrarily moves one place left or right. Thus, in the first step all candies can be moved right, and in the second step all candies can be moved back left, except for one at student B who will pass it right to C. In this way, one of the A's candies is now at C. The other direction is analogous. □

Assume n is odd. Since the distance between any two students is either clockwise or counterclockwise even, each student can repeatedly use the Lemma to send a candy to any other student in an even number of steps. Thus all arrangements are legal.

Now assume n is even. After every step, the students on even positions have at least one candy in total, and so do those on odd positions. Thus the $2\binom{3n/2-1}{n}$ arrangements in which all candies are on odd positions or all on even ones are illegal. It remains to show that all other arrangements are legal.

By the Lemma, it suffices to show that, for each $a = 1, 2, \ldots, n-1$, we can obtain at least one arrangement with exactly a candies on even positions. We start by using the Lemma to send all candies to two adjacent students A and B, with A on an even position. Suppose w.l.o.g. that, at this point, A has $a' > a$ candies. In the first step, A passes a candy to B, and B passes a candy to his other neighbor C. In the second step, A and B exchange a candy, whereas C gives his candy back to B. Now A has $a' - 1$ candy, and B has the remaining $n - a' + 1$. Repeating this procedure $a' - a$ times we reach a desired arrangement.

Therefore, the number of legal arrangements is $\binom{2n-1}{n}$ for $2 \nmid n$, and $\binom{2n-1}{n} - 2\binom{3n/2-1}{n}$ for $2 \mid n$.

Mathematical Society of Serbia

welcomes you

to the

35-th Balkan Mathematical Olympiad

to be held

in Serbia in May 2018

(the exact location and dates to be announced)

Mathematical Society of Serbia

Serbian mathematical competitions were supported by