ALGEBRA

A1. A set A is endowed with a binary operation $*$ satisfying the following four conditions:

(1) If a, b, c are elements of A, then $a*(b*c)=(a*b)*c$;
(2) If a, b, c are elements of A such that $a*c=b*c$, then $a=b$;
(3) There exists an element e of A such that $a*e=a$ for all a in A; and
(4) If a and b are distinct elements of $A\setminus\{e\}$, then $a^3*b=b^3*a^2$, where $x^k=x*x^{k-1}$ for all integers $k\geq 2$ and all x in A.

Determine the largest cardinality A may have.

Bojan Bašić, Serbia

Solution. The required maximum is 3, in which case A is necessarily a copy of the additive group of residue classes modulo 3; that is, $A=\{a,a^2,a^3=e\}$, which clearly satisfies the conditions.

We show that if $|A|\geq 3$, then A has the above form. For convenience, write $xy=x*y$ for all x, y in A. Let a and b be distinct elements of $A\setminus\{e\}$.

We first prove that either $a^2\neq a$ or $b^2\neq b$. If $a^2=a$, then $a^3=a$, and $ba=ba^2$ implies $b=ba$ by (2). Similarly, if $b^2=b$, then $b^3=b$, and $a=ab$. Hence, if $a^2=a$ and $b^2=b$, then $a=ab=a^3b=b^3a^2=ba=b$ by (4) in the middle, a contradiction.

Assume henceforth $a^2\neq a$.

Next, we show that $a^2\neq e$. Suppose $a^2=e$ and use (3) and (4) to write $ab=(ae)b=(aa^2)b=a^3b=b^3a^2=b^3e=b^3$, so $a=b^2$ by (2). Then $b^5=b^3b^2=b^3a=a^3b^2=(b^2)^3b^2=b^8$ by (4) in the middle, so $b=b^4$ by (2), whence $b=(b^2)^2=a^2=e$, a contradiction.

We now prove that $a^3=e$. Suppose $a^3\neq e$ and notice that a^2 and a^3 are distinct elements of $A\setminus\{e\}$ (otherwise, (2) would imply $a=a^2$), to use (4) and write $a^{11}=(a^3)^3a^2=(a^2)^3(a^3)^2=a^{12}$, so $a=a^2$, a contradiction.

Finally, we show that $b=a^2$, so A consists of e, a and a^2 alone. Use (4) to get $eb=a^3b=b^3a^2$ and $eb^2=a^3b^2=b^3a$. This yields $eb^2=b^3a=b^2(eb)a=b^2b^3a^2a=b^3b^2$, so $e=b^3$ by (2). Then $eb=b^3a^2=ea^2$, so $b=b^4=eb=ea^2=a^3a^2=a^2e=a^2$. This completes the proof.

COMBINATORICS

C1. A planar country has an odd number of cities separated by pairwise distinct distances. Some of these cities are connected by direct two-way flights. Each city is directly connected to exactly two other cities, and the latter are located farthest from it. Prove that, using these flights, one may go from any city to any other city.

Alexander Gaifullin, Russia

Solution. Consider the graph G whose vertices are the cities and whose edges are the direct two-way flights. We must show that G is connected.

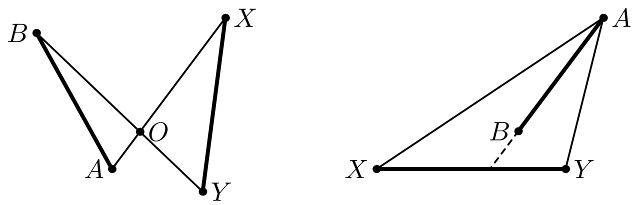
Assuming G has at least two components, we show that the cardinality of each component is even; this contradicts the fact that the total number of vertices is odd, whence the conclusion.

Lemma. *If AB and XY are edges in different components of G, then the segments AB and XY cross at an interior point.*

Proof. Two cases are to be ruled out:

(1) The lines AB and XY meet outside both open segments (AB) and (XY), possibly at an ideal point; that is, after a suitable relabelling (if necessary), $ABXY$ is a convex (possibly degenerate) quadrangle.

In this case, the segments AX and BY cross at some point O, so $AX + BY = (OA + OX) + (OB + OY) = (OA + OB) + (OX + OY) \geq AB + XY$. Consequently, either $AX \geq AB$, in which case A is connected to X, contradicting the fact that the two lie in different components of G; or $BY \geq XY$, in which case Y is connected to B, contradicting the fact that the two lie in different components of G.



(2) The lines AB and XY cross at an interior point of exactly one of open segments (AB) and (XY), say, the latter; that is, after a suitable relabelling (if necessary), the segment AB lies inside the triangle AXY.

In this case, $AB < \max(AX, AY)$, since the segment AB lies along a cevian in the triangle AXY, so A is connected to (exactly) one of X and Y, contradicting the fact that A and X, Y lie in different components of G.

Back to the problem, since every vertex has degree 2, each component of G is a circuit. Let $\mathcal{C}$ be one such and let XY be an edge of G whose endpoints do not belong to $\mathcal{C}$. By the lemma, every two consecutive vertices of $\mathcal{C}$ lie on opposite sides of the line XY, so $\mathcal{C}$ has an even number of vertices. This ends the proof.

Remarks. One possible configuration satisfying the conditions in the statement is obtained by slightly disturbing the positions of the vertices of a regular polygon with an odd number of sides, to make all pairwise distances distinct.

The condition that the total number of cities be odd is essential for the conclusion to hold. For instance, placing a pair of cities near each vertex of a square allows 'diagonal' flights only; the maximal circuits the flight graph splits into are bow ties stretched out along the diagonals. The construction generalises to any regular polygon with an even number of vertices. With some care, both 2-city clusters around a pair of opposite vertices of such a polygon can (simultaneously) be replaced by 3-city clusters; this can be done for any number of such pairs.

C2. Fix an integer $n \geq 2$ and let A be an $n \times n$ array with n cells cut out so that exactly one cell is removed out of every row and every column. A *stick* is a $1 \times k$ or $k \times 1$ subarray of A, where k is a suitable positive integer.

(a) Determine the minimal number of sticks A can be dissected into.

(b) Show that the number of ways to dissect A into a minimal number of sticks does not exceed 100^n.

Palmer Mebane and Nikolai Beluhov

Solution 1. (a) The required minimum is $2n-2$ and is achieved, for instance, by dissecting A along all horizontal (or vertical) grid lines.

By *holes* we mean the cells which are cut out from the board. The *cross* of a hole in A is the union of the row and the column through that hole.

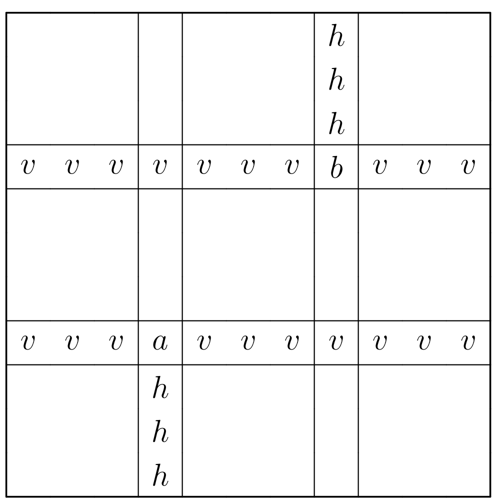
Consider a dissection of A into $2n-2$ or fewer sticks. Horizontal sticks are all labelled h, and vertical sticks are labelled v; 1×1 sticks are both horizontal and vertical, and labelled arbitrarily. Each cell of A inherits the label of the unique containing stick.

Assign each stick in the dissection to the cross of the unique hole on its row, if the stick is horizontal; on its column, if the stick is vertical.

Since there are at most $2n-2$ sticks and exactly n crosses, either there is a cross assigned to no stick in the dissection, or there are two crosses each of which is assigned to exactly one stick in the dissection.

Case 1: There is a cross assigned to no stick in the dissection. In this case, the vertical arms of the cross are both all-h, and the horizontal arms are both all-v, so no stick in the dissection covers more than one of the $2n-2$ cells along the arms of the cross. Consequently, there are at least $2n-2$ sticks.

Case 2: There are two crosses each of which is assigned to exactly one stick in the dissection. Let the crosses be c and d, centred at $a = (x_a, y_a)$ and $b = (x_b, y_b)$, respectively, and assume, without loss of generality, $x_a < x_b$ and $y_a < y_b$. The sticks covering the cells (x_a, y_b) and (x_b, y_a) have like labels, for otherwise one of the two crosses would be assigned to at least two sticks. Say the common label is v, to infer that the lower (respectively, upper) arm of c (respectively, d) is all-h, and the horizontal arms of both crosses are all-v, as illustrated below.



Each of the rows between the rows of a and b, that is, rows y_a+1, y_a+2, ..., y_b-1, contains a hole. The column of each such hole contains at least two v-sticks. All other columns contain at least one v-stick each. In addition, all rows below a and all rows above b contain at least one h-stick each. This amounts to a total of at least $2(y_b-y_a-1)+(n-y_b+y_a+1)+(n-y_b)+(y_a-1)=2n-2$ sticks.

(b) To provide an upper bound for the number of minimal dissections of A into sticks, the lemma below will be applied to the two cases in the solution to part **(a)**.

Lemma. *Consider a $p\times q$ array B with some cells removed, at most one from each row and each column; assume the cell in the lower left corner is among the removed cells. Then there are at most 2^{p+q} dissections of B into $p-1$ vertical sticks, each of which has a cell on the bottom row, and $q-1$ horizontal sticks, each of which has a cell on the leftmost column.*

Proof. Locate the holes at (x_i, y_i), where i runs through some initial segment of the positive integers.

If $x_i < x_j$ and $y_i > y_j$ for some indices i and j, the cell (x_j, y_i) is covered by no stick having a cell on the bottom row or on the leftmost column, so the number of dissections satisfying the condition in the lemma is zero.

Otherwise, label the holes so that $x_i < x_{i+1}$ and $y_i < y_{i+1}$ and notice that the union of all horizontal sticks in a dissection satisfying the conditions in the lemma is separated from the union of all vertical sticks in that dissection by a broken line running up-and-to-the-right along grid lines from the lower left corner of B to the upper right corner of B.

Since each such path determines at most one dissection satisfying the condition in the lemma, the number of possible dissections does not exceed the number of such paths, which is $\binom{p+q}{p}\leq 2^{p+q}$.

Back to the problem, consider dissections $\mathcal{D}$ of A into $2n-2$ sticks, along with the two cases in the solution to part **(a)**.

Case 1: There is a cross c, assigned to no stick in $\mathcal{D}$; let a be the hole at the center of c. The cross c subdivides A into four quadrants. Extend each of these quadrants by adding the two arms of c that bound it, along with the hole a, to obtain an array as described in the lemma. Since $|\mathcal{D}|=2n-2$, each stick in $\mathcal{D}$ contains exactly one cell of the cross c. Therefore, the restriction of $\mathcal{D}$ to each extended quadrant satisfies the condition in the lemma, so there are at most 2^{4n+4} such dissections involving c; and since there are n options for the centre of c, the number of these dissections does not exceed $2^{4n+4}n$.

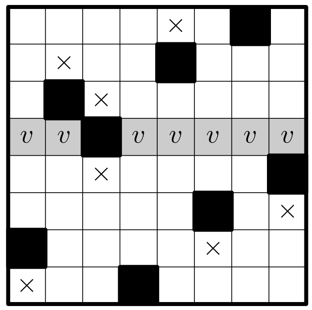
Case 2: There are two crosses each of which is assigned to exactly one stick in $\mathcal{D}$. Use the notation, conventions and choices in the solution to part **(a)**. Since $|\mathcal{D}|=2n-2$, the evaluation in part **(a)** exhausts the sticks in $\mathcal{D}$. So the restriction of $\mathcal{D}$ to the strip flanked by the two rows through a and b, respectively, consists of vertical sticks only; and the restrictions of $\mathcal{D}$ to the extensions of the upper (respectively, lower) quadrants around b (respectively, a) satisfy the condition in the lemma. It follows that there are at most 2^{4n+2} such dissections involving c and d. Since there are $\binom{n}{2}=n(n-1)/2$ options for the centres of c and d, and the sticks covering the cells (x_a, y_b) and (x_b, y_a) may be labelled two ways (recall that they have like labels), the number of these dissections does not exceed $2^{4n+2}n(n-1)$.

Consequently, the total number of minimal dissections of A into sticks does not exceed $2^{4n+2}n(n+3)$; the latter is less than 100^n, provided that $n\geq 3$, and the case $n=2$ is clear.

Solution 2. (*Ilya Bogdanov*) **(a)** Cutting the board along the horizontal grid lines produces a dissection into $2n-2$ sticks. So we are left to prove that this is the minimal possible number of sticks in a dissection.

Call a stick *vertical* if it is contained in some column, and *horizontal* if it is contained in some row; 1×1 sticks may be called arbitrarily, but any of them is supposed to have only one direction. Assign to each vertical/horizontal stick the column/row it is contained in. If each row and each column is assigned to some stick, then there are at least $2n$ sticks, which is even more than we want. Thus we assume, without loss of generality, that some *exceptional* row R is not assigned to any stick. This means that all $n-1$ existing cells in R belong to $n-1$ distinct vertical sticks; call these sticks *central*.

Now we mark $n-1$ cells on the board in the following manner. ($\downarrow$) For each hole c below R, we mark the cell just under c; ($\uparrow$) for each hole c above R, we mark the cell just above c; and ($\bullet$) for the hole r in R, we mark both the cell just above it and just below it. We have described $n+1$ cells, but exactly two of them are out of the board; so $n-1$ cells are marked within the board. A sample marking is shown in the figure below, where the marked cells are crossed.



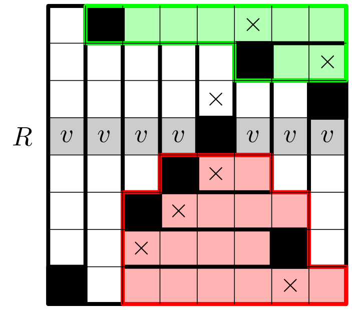
Notice that all the marked cells lie in different rows, and all of them are marked in different columns, except for those two marked for ($\bullet$); but the latter two have a hole r between them. So no two marked cells may belong to the same stick. Moreover, none of them lies in a central stick, since the marked cells are separated from R by the holes. Thus the marked cells should be covered by $n-1$ different sticks (call them *border*) which are distinct from the central sticks. This shows that there are at least $(n-1)+(n-1)=2n-2$ distinct sticks, as desired.

(b) Consider any dissection into exactly $2n-2$ sticks. It contains an exceptional row or an exceptional column; we consider only the first option (so we will need to double our bound at the end) and again denote one exceptional row by R. Now we may (and will) assume that all 1×1 sticks are vertical.

The dissection contains exactly $n-1$ central (vertical) sticks and exactly $n-1$ border sticks. Such a dissection is completely determined by the arrangement of horizontal (border) sticks. Indeed, if we delete all horizontal sticks from the board, the remaining part can be dissected into the smallest possible number of vertical sticks in the unique way (along all vertical grid lines).

Thus, we are to bound the number of arrangements of horizontal sticks which may appear in our dissections. For this purpose, let us discover some properties of such arrangements. Notice first that each row contains at most one horizontal stick, since each such stick should contain a marked cell.

For every horizontal stick B, let b be the hole in its row; then we say that the *extended stick* B' of B is either $B\cup\{b\}$ (if B and b share a side) or B (otherwise). Let B be any horizontal stick below R. We claim that the cells just below B' all lie in one extended horizontal stick; this yields that the extended sticks are arranged in two non-shrinking stacks above and below R, as illustrated in the figure below.



Let x be any of cells just below B (there are at least two such). If x lies in some vertical stick S, then S is separated from R by B. Thus S should contain a marked cell, and this marked cell should be the topmost one, i.e., x. This cannot happen, since in this case the cell just above x should be missing, but in fact it belongs to B. Therefore, x either lies in a horizontal stick or is missing. Since there are at least two options for x, the row below B contains a horizontal stick, which should contain a marked cell (lying below the hole b in the row of B). Hence all cells below B' are either missing or contained in (the same) horizontal stick. The claim is proved.

Assume now that there are h horizontal sticks below R; then they are situated in the lowest h rows. Moreover, if the extended stick in the ith row occupies the columns from a_i to b_i, then our claim yields $1 \leq a_1 \leq a_2 \leq \cdots \leq a_h \leq b_h \leq \cdots \leq b_1 \leq n$. Such a collection $(a_1, \ldots, a_h, b_h, \ldots, b_1)$ can be chosen in $\binom{n+2h-1}{2h}$ ways, and it completely determines the arrangement of horizontal sticks below R. So the number of such arrangements is at most

$$\sum_{h=0}^{n-1} \binom{n+2h-1}{2h} \leq \sum_{k=0}^{2n-2} \binom{n+k-1}{k} = \binom{3n-2}{n} \leq 2^{3n-2}.$$

The same bound holds for the number of arrangements of horizontal sticks above R. Hence the number of minimal dissections (recall that we should double the bound) does not exceed $2 \cdot (2^{3n-2})^2 = 2^{6n-3}$, as required.

Solution 3 for (a). It is sufficient to show that there are $2n-2$ cells in A, no two of which may be contained in the same stick.

To this end, consider the bipartite graph G with parts G_h and G_v, where the vertices in G_h (respectively, G_v) are the $2n-2$ maximal sticks A is dissected into by all horizontal (respectively, vertical) grid lines, two sticks being joined by an edge in G if and only if they share a cell.

We show that G admits a perfect matching by proving that it fulfils the condition in Hall's theorem; the $2n-2$ cells corresponding to the edges of this matching form the desired set. By symmetry, it is sufficient to show that every subset S of G_h has at least $|S|$ neighbours (in G_v, of course).

Let L be the set of all sticks in S that contain a cell in the leftmost column of A, and let R be the set of all sticks in S that contain a cell in the rightmost column of A; let ℓ be the length of the longest stick in L (zero if L is empty), and let r be the length of the longest stick in R (zero if R is empty).

Since every row of A contains exactly one hole, L and R partition S; and since every column of A contains exactly one hole, neither L nor R contains two sticks of the same size, so $\ell \geq |L|$ and $r \geq |R|$, whence $\ell + r \geq |L| + |R| = |S|$.

If $\ell + r \leq n$, we are done, since there are at least $\ell + r \geq |S|$ vertical sticks covering the cells of the longest sticks in L and R. So let $\ell + r > n$, in which case the sticks in S span all n columns, and notice that we are again done if $|S| \leq n$, to assume further $|S| > n$.

Let $S' = G_h \smallsetminus S$, let T be set of all neighbours of S, and let $T' = G_v \smallsetminus T$. Since the sticks in S span all n columns, $|T| \geq n$, so $|T'| \leq n-2$. Transposition of the above argument (replace S by T'), shows that $|T'| \leq |S'|$, so $|S| \leq |T|$.

Remarks. The authors suspect that the number of ways to dissect A into $2n-2$ sticks does not exceed 4^{n-2}, equality being achieved exactly when the holes are all strung along a main diagonal.

There are several ways of improving the estimate in Solution 2. For instance, the total height of the "stacks" of horizontal sticks above R and below R does not exceed $n-1$; this leads to an estimate of order 8^n, up to a polynomial factor in n.

GEOMETRY

G1. Let $ABCD$ be a trapezium, $AD \parallel BC$, and let E and F be points on the sides AB and CD, respectively. The circumcircle of the triangle AEF meets the line AD again at A_1, and the circumcircle of the triangle CEF meets the line BC again at C_1. Prove that the lines A_1C_1, BD, and EF are concurrent.

Alexander Kuznetsov, Russia

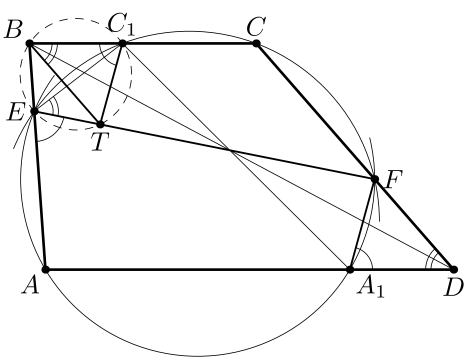
Solution 1. Let T be the second meeting point of the circumcircle of the triangle C_1BE with the line EF. Then

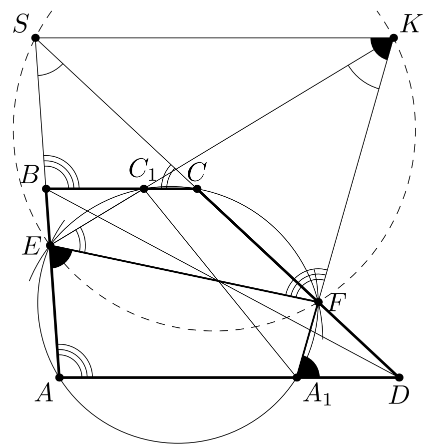
$$\angle(TC_1, C_1B) = \angle(TE, EB) = \angle(FE, EA) = \angle(FA_1, A_1A),$$

so $TC_1 \parallel FA_1$ (since $C_1B \parallel A_1A$). Similarly,

$$\angle(TB, BC_1) = \angle(TE, EC_1) = \angle(FE, EC_1) = \angle(FC, CC_1) = \angle(FD, DA_1),$$

so $TB \parallel FD$. Thus the corresponding sides of the triangles C_1BT and A_1DF are parallel; notice also that the vectors $\overline{C_1B}$ and $\overline{A_1D}$ are counter-directed. Therefore, these triangles are homothetical with a negative coefficient. The center of the respective homothety O lies on the lines BD, A_1C_1, and TF (which coincides with EF), so these lines are concurrent.





Solution 2. Let $S = AB \cap CD$, $K = C_1E \cap A_1F$. We have

$$\begin{aligned}\angle(EK, FK) &= \angle(EC_1, EF) + \angle(EF, A_1F) = \angle(CC_1, CF) + \angle(EA, AA_1)\\ &= \angle(BC, FS) + \angle(ES, BC) = \angle(ES, FS).\end{aligned}$$

This means that the points E, F, S, and K are concyclic. Thus $\angle(KS, KF) = \angle(ES, EF) = \angle(EA, EF) = \angle(A_1A, A_1F)$, which means that $KS \parallel BC \parallel AD$.

Now apply the Desargues' theorem to the triangles C_1BE and A_1DF. The lines C_1B and A_1D meet at an ideal point X which is collinear to $S = BE \cap DF$ and $K = C_1E \cap A_1F$, as was shown above. This means that the lines C_1A_1, BD, and EF are concurrent, as required. (Since the segments A_1C_1 and BD meet, the concurrency point is not ideal.)

G2. Let ABC be a triangle. Consider the circle ω_B internally tangent to the sides BC and BA, and to the circumcircle of the triangle ABC; let P be the point of contact of the two circles. Similarly, consider the circle ω_C internally tangent to the sides CB and CA, and to the circumcircle of the triangle ABC; let Q be the point of contact of the two circles. Show that the incentre of the triangle ABC lies on the segment PQ if and only if $AB + AC = 3BC$.

Luis Eduardo García Hernández, Mexico

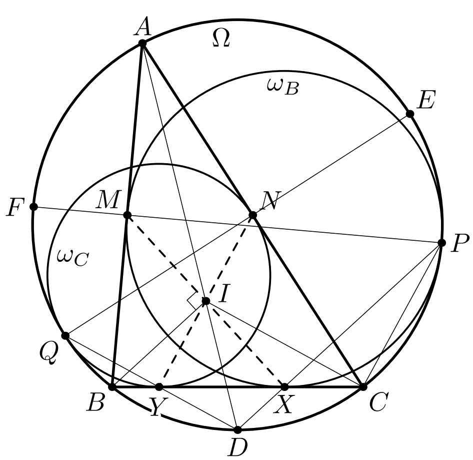
Solution 1. Let I, respectively Ω, be the incentre, respectively circumcircle, of the triangle ABC. Let the B-mixtilinear circle ω_B touch the sides BC and BA at X and M, respectively, and let the C-mixtilinear circle ω_C touch the sides CB and CA at Y and N, respectively. Let D, E, and F be respectively the midpoints of the arcs BC, AC, and AB of Ω not containing other vertices of the triangle ABC. We start by collecting some useful (and known) information about the circles ω_B and ω_C.

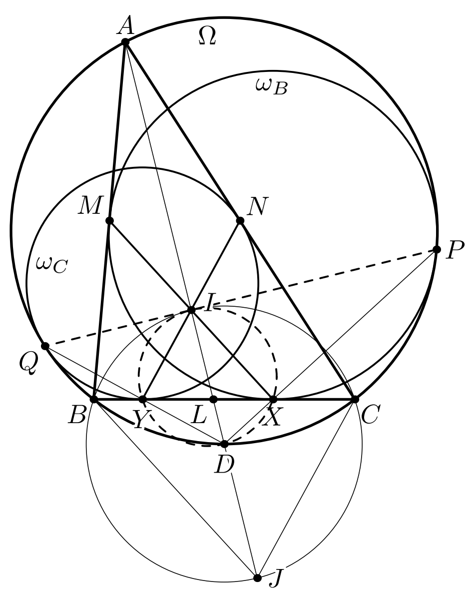
Lemma 1. *Each of the triples (P, M, F), (Q, N, E), (P, X, D), and (Q, Y, D) consists of three collinear points.*

Proof. The homothety centred at P and mapping ω_B into Ω maps the point M to a point on the arc AB of Ω which belongs to the tangent parallel to AB, i.e., M maps to F. Thus the points P, M, and F are collinear. The proof for the other triples is similar.

Lemma 2. *The points M, I, X (respectively, N, I, Y) are collinear, and MX (respectively, NY) is perpendicular to the angle bisectrix BI (respectively, CI).*

Proof. By Lemma 1, we have $M = AB \cap FP$, $X = BC \cap PD$, and obviously $I = CF \cap DA$. Application of Pascal's theorem to the hexagon $ABCFPD$ shows that the three points are collinear. Finally, the internal angle bisectrix BI of the triangle BMX is perpendicular to MX, since $BM = BX$.





Now we turn to the solution. Let AL, L on BC, be the the internal angle bisectrix of the triangle ABC, and let J be the A-excentre. It is well known that the points B, I, C, and J lie on a circle centred at D. The inversion with respect to this circle preserves B, I, and C, and it swaps Ω and BC; so by Lemma 1 it maps P, I, and Q to X, I, and Y, respectively. Therefore, I lies on PQ if and only if $IXDY$ is cyclic.

Due to Lemma 2, the corresponding sides of the triangles IXY and JBC are parallel, so there exists a homothety (centred at L) mapping one to the other. This homothety maps D to some point D', and the quadrangle $IXDY$ is cyclic if and only if the quadrangle $JBD'C$ is cyclic, i.e. if and only if $D' = I$, or $LI^2 = LD \cdot LJ$. Since D is the midpoint of IJ, this is equivalent to $LJ = 2LI$, which is the case if and only if $r_a = 2r$ (standard notation). To conclude, refer to the standard formulae $r_a = S/(s-a)$ and $r = S/s$, where S and s are the area and the semiperimeter of the triangle ABC, respectively.

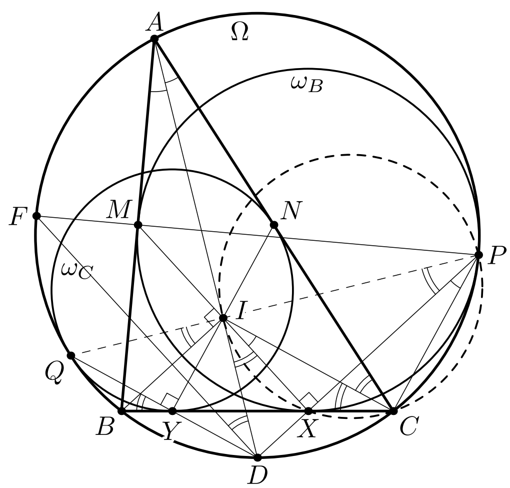
Solution 2. Use the notation in Solution 1 and assume the two lemmata. Now we prove separately that each of the two conditions implies the other. To start, we collect even more (known) information on the mixtilinear circles.

Lemma 3. *The quadrangle $CXIP$ (respectively, $BYIQ$) is cyclic.*

Proof. Let $\angle BAC = 2\alpha$ and recall that $\angle BIC = 90^\circ + \alpha$. By Lemma 2, $\angle BIX = 90^\circ$, so $\angle CIX = \angle BIC - \angle BIX = \alpha$. Since $\angle CPX = \angle CPD = \alpha$, the conclusion follows.

Lemma 4. *The triangles DIP (respectively, DIQ) and DXI (respectively, DYI) are similar.*

Proof. Let $\angle ACB = 2\gamma$ and refer to Lemma 3 to write $\angle DPI = \angle XPI = \angle ICX = \gamma$. Notice that $DF \parallel XM$ by means of the homothety centred at P, to deduce that $\angle DIX = \angle IDF = \gamma$. Thus, $\angle DPI = \angle DIX$, and since the two triangles under consideration share the internal angle at D, they are similar.



Let M' and N' be the midpoints of the sides AB and AC, respectively. Reflect M', respectively N', in the angle bisectrix BI, respectively CI, to obtain the point X', respectively Y'. Clearly, $BX' = BM' = BA/2$ and $CY' = CN' = CA/2$, so

$$Y'X' = BX' - BY' = BX' - (BC - Y'C) = AB/2 - BC + AC/2. \qquad (*)$$

Thus the relation $AB + AC = 3BC$ is equivalent to $Y'X' = BC/2 = M'N'$ (notice that the right-hand part in $(*)$ can never reach $-BC/2$ by the triangle inequality).

Assume now the relation $X'Y' = M'N'$; then $M'N'X'Y'$ is a parallelogram, so the segments $M'X'$ and $N'Y'$ share a midpoint. Since the triangles $BM'X'$ and $CN'Y'$ are isosceles, this common midpoint lies on the angle bisectrices BI and CI, so $M'X' \cap N'Y' = I$. Refer to Lemma 2 to deduce that $M' = M$ and $X' = X$.

Thus, ω_B touches the chord BA of Ω at the midpoint. So the line FMP is perpendicular to AB and therefore contains diameters FP and MP of Ω and ω_B, respectively. Hence $\angle IXP = \angle MXP = 90^\circ$, so the angle DXI is right.

Refer now to Lemma 4 to deduce that the angle DIP is also right. Similarly, the angle DIQ is right, so the points P, I, Q are collinear.

Conversely, assume that the points P, I, and Q are collinear. Refer to Lemma 4 to write $DX \cdot DP = DI^2 = DY \cdot DQ$ and infer thereby that the quadrangle $XPQY$ is cyclic, so $\angle QPX = \angle QYB$.

By Lemma 3, $\angle QYB = \angle QIB$, so $\angle QPX = \angle QIB$. The latter along with collinearity of P, I, Q show that BI and DP are parallel, so PX and MX are perpendicular by Lemma 2.

Similarly, QY and NY are perpendicular, so MP and NQ are diameters of ω_B and ω_C, respectively. It follows that M and N are the midpoints of AB and AC, so $M' = M$, $X' = X$, $N' = N$, $Y' = Y$. By Lemma 2, the quadrangle $MNXY$ is a parallelogram with centre I, so $M'N' = X'Y'$. This ends the proof.

G3. Let $ABCD$ be a convex quadrangle and let P and Q be variable points inside this quadrangle so that $\angle APB = \angle CPD = \angle AQB = \angle CQD$. Prove that the lines PQ obtained in this way all pass through a fixed point, or they are all parallel.

Sergey Berlov, Russia

Solution. Assume first that there exists a point M such that the triangles AMC and DMB are similar and have opposite orientations. We claim that every possible line PQ passes through M.

By the condition in the statement, the points A, B, P, and Q lie on some circle ω_1, and the points C, D, P, and Q lie on some circle ω_2. The line PQ is the radical axis of these two circles; so we need to prove that M also lies on this radical axis.

Let MA meet again ω_1 at A', and let MD meet again ω_2 at D' (if, say, MA is tangent to ω_1, then $A' = A$). Then

$$\angle(MA', A'B) = \angle(AA', A'B) = \angle(AP, PB) = \angle(CP, PD) = \angle(CD', D'D) = \angle(CD', D'M)$$

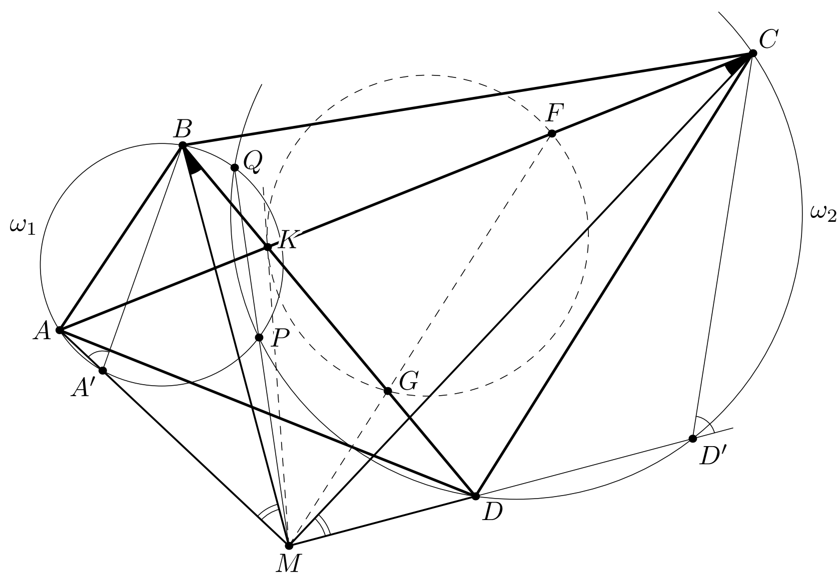
and

$$\begin{aligned}\angle(MA', MB) &= \angle(MA, MB) = \angle(MA, MC) + \angle(MC, MB) \\ &= \angle(MB, MD) + \angle(MC, MB) = \angle(MC, MD) = \angle(MC, MD').\end{aligned}$$

This means that the triangles $MA'B$ and $MD'C$ are similar and have opposite orientations. Hence

$$\frac{MA'}{MD'} = \frac{MB}{MC} = \frac{MD}{MA},$$

so $MA{\cdot}MA' = MD{\cdot}MD'$. Since $\angle(\overline{MA'}, \overline{MC}) = \angle(\overline{MB}, \overline{MD'})$ and $\angle(\overline{MA}, \overline{MC}) = \angle(\overline{MB}, \overline{MD})$, the point M lies either outside both segments AA' and BB', or inside both. Thus the relation obtained means exactly that M lies on the radical axis of ω_1 and ω_2.



It remains to check whether such a point M exists, and deal with the exceptional cases. This can be done in several different ways, e.g., by investigating the orientation-changing similarity transformation mapping A to D and B to C. We present a more geometrical proof.

Set $K = AC \cap BD$. Choose the points F and G on the diagonals AC and BD, respectively, so that the configurations (A, F, K, C) and (D, K, G, B) are similar. If $AD \nparallel BC$, then we have $F \neq K \neq G$. Due to similarity, it suffices to find the point M such that the triangles MKF and MGK are similar and have opposite orientations. If $KF \neq KG$, one may choose M to be the point where FG meets the tangent at K to the circumcircle of the triangle FGK; so in this case we are done.

The exceptional cases are $AD \parallel BC$ and $KF = KG$ (which means that $AC = BD$ due to similarity). These cases may be considered as limit cases: in the former case, M comes to K (and this can be dealt with along the above lines); in the latter, M becomes an ideal point (so the lines PQ are all parallel to each other). In fact, both cases can be dealt with independently (and are relatively easy).

It is worth mentioning that the triangles $MA'B$ and $MD'C$ used in the solution may become degenerate. It happens when $M = AB \cap CD$, i.e. when the quadrilateral $ABCD$ is cyclic. But in this case M is the radical center of the circles ω_1, ω_2, and $(ABCD)$, so in this case the claim holds. This may also happen when $M = A' = D'$; but then M coincides with one of P and Q, and the claim is also trivial.

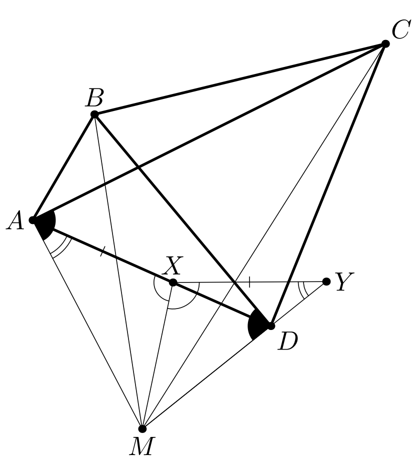
Remarks. Let us sketch some other approaches which may lead to different solutions (and also to different versions of the same solution).

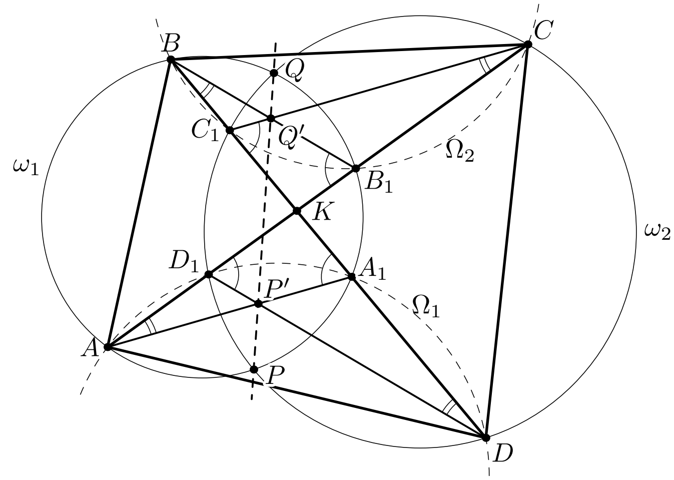
1. The required point M can be described in different ways; we provide one more such. Choose a point X on the segment AD so that $AX/XD = AC/BD$. Next, choose a point Y so that $\angle(XY, XD) = \angle(DB, DA) + \angle(AC, AD)$ and $YX = XA$. Let the internal bisectrix of the angle AXY meet DY at M. Then M is a required point.

To see this, notice that the points A and Y are symmetric to each other with respect to MX. Therefore,

$$\frac{MD}{MA} = \frac{MD}{MY} = \frac{DX}{XY} = \frac{DX}{XA} = \frac{BD}{AC},$$

and $\angle(AM, AC) = \angle(AM, AD) + \angle(AD, AC) = \angle(YX, YM) + \angle(AD, AC) = \angle(AD, AC) + \angle(XY, XD) + \angle(DX, DY) = \angle(DB, DA) + \angle(DA, DM) = \angle(DB, DM)$. These relations, along with the position of M, yield that the triangles MAC and MDB are similar and oriented differently.





2. Denote by A_1 and B_1 the second meeting points of ω_1 with BD and AC, respectively; denote by C_1 and D_1 the second meeting points of ω_2 with BD and AC, respectively. Then

$$\angle(CC_1, C_1D) = \angle(CD_1, D_1D) = \angle(AA_1, A_1B) = \angle(AB_1, B_1B) = \angle(AP, PB).$$

Thus $AA_1 \parallel CC_1$ and $BB_1 \parallel DD_1$.

Moreover, one sees that the quadrilaterals AD_1A_1D and BC_1B_1C are inscribed into some circles Ω_1 and Ω_2. Thus the points $P' = AA_1 \cap DD_1$ and $Q' = BB_1 \cap CC_1$ are the radical centers of $(\omega_1, \omega_2, \Omega_1)$ and $(\omega_1, \omega_2, \Omega_2)$, so the lines PQ and $P'Q'$ coincide. Hereafter we deal with the line $P'Q'$ defined uniquely in terms of an angle $\alpha = \angle(A_1A, A_1B) = \angle(B_1A, B_1B) = \angle(C_1C, C_1D) = \angle(D_1C, D_1D)$.

Let us fix the points A and D and the value of α, and move B and C along BD and AC so that the ratio AC/BD remains constant. Then the point P' remains constant, while the point Q' moves along some line; it comes to P' when B comes to D, so this line is $P'Q'$. Thus *all* the lines of the form $P'Q'$ remain constant during such movement.

Apply such movement firstly to put B into K. Then fix B and C, and apply a similar movement to A and D passing A to K as well. The quadrilateral $ABCD$ comes to a degenerate quadrilateral $KKFG$ from the Solution. Thus it suffices to solve the problem for this degenerate quadrilateral.

Now the point M becomes more visible (from some particular cases), and it is easier to prove the problem statement for this point.

3. Yet another way of defining the point M. Let Γ_1 and Γ_2 be the Apollonius circles constructed for the segments AD and CB, respectively, and a common ratio AC/BD (by definition, if $X \in \Gamma_1$ and $Y \in \Gamma_2$, then $AX/XD = CY/YB = AC/BD$). One of their common points is the (unique) center of rotational homothety mapping AC to DB. The other common point can serve as M, if it exists.

So, while dealing with this approach, one should elaborate on the following exceptional cases: (i) the circles Γ_1 and Γ_2 are tangent, and (ii) they degenerate to lines (which happens when $AC = BD$).

NUMBER THEORY

N1. Let $S(k)$ denote the sum of digits in the decimal representation of a positive integer k. Prove that there exists a positive integer k that does not contain the digit 9 in its decimal representation, such that $S(2^{24^{2017}}k) = S(k)$.

Bojan Bašić, Serbia

Solution 1. Set $N = 2^{24^{2017}}$. A routine check shows that N ends with digit 6. Define $f(k) = S(Nk) - S(k)$, and let A be the range of f on positive integers with no 9s in the decimal representation. Notice that $A + A \subseteq A$, since for every positive integers a and b we have $f(10^s a + b) = f(a) + f(b)$, provided that s is large enough. Thus, in order to solve the problem, it suffices to find positive integers m and ℓ with no 9s in their decimal representations, satisfying $f(m) \leq 0$ and $f(\ell) > 0$, since then $0 = \underbrace{f(m) + \cdots + f(m)}_{f(\ell)} + \underbrace{f(\ell) + \cdots + f(\ell)}_{|f(m)|} \in A$. Clearly, the value $\ell = 1$ fits, so the problem is reduced to finding a suitable m.

In order to preform this, it suffices to prove that for every positive integer d there is a d-digit number m_d such that each of the last d digits in Nm_d is strictly smaller than the corresponding digit in m_d. Indeed, choosing D so that $10^D > N$, one may take $m = m_{10D}$. To check this choice, notice that Nm contains at most $11D$ digits, and the sum of its $10D$ last digits does not exceed $S(m) - 10D$, while the sum of the remaining digits is at most $9D$. Thus $f(m) \leq -10D + 9D = -D < 0$.

Finally, we prove the desired claim by the induction on d. In the base case $d = 1$ one may set $m_1 = 7$, since $7N$ ends with 2. In order to prove the step, assume that the number m_{d-1} has been already constructed, and denote by j the dth tail digit of Nm_{d-1}.

If $j \leq 4$, then we set $m_d = \overline{7m_{d-1}} = 7 \cdot 10^{d-1} + m_{d-1}$. Then the $d-1$ tail digits in m_d and Nm_d are the same as those in m_{d-1} and Nm_{d-1}, respectively. Next, their dth tail digits are 7 and $2 + j < 7$, respectively; so m_d satisfies all the requirements imposed on it.

Otherwise, if $5 \leq j \geq 9$, we set $m_d = \overline{8m_{d-1}} = 8 \cdot 10^{d-1} + m_{d-1}$. A similar argument applies, since the dth tail digits in m_d and Nm_f in this case are 8 and $j - 2 < 8$, respectively. This completes the proof.

Solution 2. As in Solution 1, we reduce the problem to finding a positive integer m with no 9s in its decimal representation such that $S(m) > S(Nm)$. Now we provide a different choice for m. Namely, we show that

$$m_d = \underbrace{\overline{8787\ldots87}}_{d \text{ times}} = 87 \cdot \frac{10^{2d} - 1}{99}$$

fits, provided that d is large enough.

A routine check shows that $N \equiv 82 \pmod{99}$, so $87N \equiv 6 \pmod{99}$. Fix a number q such that $10^{2q} > 87N$. Then

$$Nm_d = \frac{87N \cdot 10^{2d} - 87N}{99} = \frac{87N - 6}{99} \cdot 10^{2d} + 6 \cdot \frac{10^{2(d-q)} - 1}{99} \cdot 10^{2q} + \frac{6 \cdot 10^{2q} - 87N}{99},$$

where all three summands are positive integers. Thus the decimal representation of Nm contains a run of $d-q$ copies of $\overline{06}$, flanked by the decimal representations of the fixed numbers $\frac{87N-6}{99}$ and $\frac{6 \cdot 10^{2q} - 87N}{99}$. Therefore, $S(Nm_d) = 6(d-q) + \text{const}$, so $S(m_d) = 87d > S(Nm_d)$ if d is sufficiently large. This ends the proof.

Remark 1. The author mentions that the assertion in the problem actually holds for each positive integer N; the sketch of the proof follows.

The problem is easily reducible to the case when N does not end with 0. If N ends with the digit $a \neq 1$, the proof is a straightforward generalization of the one given above: for every possible value of j, one may find a suitable value for the first digit of m_d; moreover, this digit can always be chosen from the set $\{6,7,8\}$. The proof for the case $a=1$ is based on a similar idea, but it is much more technically involved.

Remark 2. The choice of the number $N = 2^{24^{2017}}$ is motivated by two observations. On one hand, in view of the above remark, the rightmost nonzero digit of N should be greater than 1, in order to avoid technicalities. On the other hand, the number N is chosen to be large enough, and also satisfy $N \equiv 1 \pmod 9$.

The other residue classes modulo 9 admit a simpler solution. If $n \equiv 0 \pmod 3$, then a sufficiently long number of the form $m = \overline{33\ldots36}$ fits. On the other hand, if $n \not\equiv 0, 1 \pmod 9$, then a sufficiently long number of the form $m = \overline{888\ldots888}$ fits.

Finally, notice that omission of the condition of k not containing a digit 9 would also admit a much simpler problem. If $N \mid 10^n$ for some n (as in the problem statement), then one may simply set $m = 10^n/N$. More generally, for an arbitrary N it suffices to choose m such that Nm is the least multiple of N having exactly d digits, for an appropriately chosen value of d.

N2. Let x, y, and k be three positive integers. Prove that there exist a positive integer N and a set of $k+1$ positive integers $\{b_0, b_1, b_2, \ldots, b_k\}$ such that for every $i = 0, 1, \ldots, k$ the b_i-ary expansion of N is a 3-digit palindrome, and the b_0-ary expansion is exactly $\overline{xyx}$.

Bojan Bašić, Serbia

Solution. We provide a general construction which may be specified in many different ways.

To start, we choose some distinct positive integers $d_1, d_2, \ldots, d_k$ satisfying the following conditions: (i) each of them is coprime with x; (ii) $d_i > 100(x+y)$ for all i; and (iii) the number $L = \operatorname{lcm}(d_1+1, d_2+1, \ldots, d_k+1)$ is coprime with $D = d_1 d_2 \ldots d_k$. Such k-tuples of numbers exist; for instance, one may choose $d_i = 2^{p_i} - 1$, where p_i are distinct large primes. Another option is to choose some large odd d_1 coprime with x, and then proceed inductively by setting $d_{i+1} = x d_1(d_1+1) d_2(d_2+1) \ldots d_i(d_i+1) + 1$.

Since $\gcd(xL, D) = 1$, the multiples of xL represent all residue classes modulo D; in particular, there exists a positive integer m such that $xLm \equiv -y \pmod{D}$. Set $B = Lm$.

Finally, we define

$$b_0 = B, \quad c_i = \frac{B}{d_i+1}, \quad b_i = d_i c_i = B - c_i, \quad \text{and} \quad N = \overline{xyx}_B = x(B^2+1) + yB.$$

We claim that these values satisfy all the requirements; moreover, for every $i = 1, \ldots, k$ the b_i-ary representation of N has three digits, starting and ending with x.

To show this, notice that for every $i \in \{1, 2, \ldots, k\}$ we have

$$N = x(d_i+1)^2 c_i^2 + y(d_i+1)c_i + x = \big(x \cdot (d_i c_i)^2 + x\big) + (2c_i x + y) \cdot (d_i c_i) + (x c_i^2 + y c_i).$$

Clearly, x is a b_i-ary digit, since $b_i \geq d_i > x$. Denote $\ell_i = (2c_i x + y) \cdot (d_i c_i)$ and $r_i = x c_i^2 + y c_i$. Now it remains to prove that the number $\ell_i + r_i$ is divisible by $b_i = d_i c_i$, and that $\ell_i + r_i < b_i^2$.

Notice that

$$x c_i + y = \frac{xB + y(d_i+1)}{d_i+1} = \frac{(xB+y) + d_i y}{d_i+1}.$$

The numerator of the last fraction is divisible by d_i (due to the choice of B), and the denominator is coprime with d_i; so $d_i \mid x c_i + y$. Thus $b_i = d_i c_i \mid r_i$; set $r_i = b_i q_i$. Notice that $q_i \leq x c_i + y$.

Hence, $\ell_i + r_i = b_i(2c_i x + y + q_i)$, where $0 \leq 2c_i x + y + q_i \leq 3c_i x + 2y < 100 c_i(x+y) < c_i d_i = b_i$. Thus the b_i-ary expansion of N is a palindrome of the form $N = \overline{x(2c_i x + y + q_i)x}_{b_i}$.