The 9^{th} Romanian Master of Mathematics Competition

Day 1: Friday, February 24, 2017, Bucharest

Language: English

Problem 1. **(a)** Prove that every positive integer n can be written uniquely in the form

$$n = \sum_{j=1}^{2k+1} (-1)^{j-1} 2^{m_j},$$

where $k \geq 0$ and $0 \leq m_1 < m_2 < \cdots < m_{2k+1}$ are integers.
This number k is called the *weight* of n.

(b) Find (in closed form) the difference between the number of positive integers at most 2^{2017} with even weight and the number of positive integers at most 2^{2017} with odd weight.

Problem 2. Determine all positive integers n satisfying the following condition: for every monic polynomial P of degree at most n with integer coefficients, there exists a positive integer $k \leq n$, and $k+1$ distinct integers $x_1, x_2, \ldots, x_{k+1}$ such that

$$P(x_1) + P(x_2) + \cdots + P(x_k) = P(x_{k+1}).$$

Note. A polynomial is *monic* if the coefficient of the highest power is one.

Problem 3. Let n be an integer greater than 1 and let X be an n-element set. A non-empty collection of subsets $A_1, \ldots, A_k$ of X is *tight* if the union $A_1 \cup \cdots \cup A_k$ is a proper subset of X and no element of X lies in exactly one of the A_is. Find the largest cardinality of a collection of proper non-empty subsets of X, no non-empty subcollection of which is tight.

Note. A subset A of X is *proper* if $A \neq X$. The sets in a collection are assumed to be distinct. The whole collection is assumed to be a subcollection.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.

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Day 2: Saturday, February 25, 2017, Bucharest

Language: English

Problem 4. In the Cartesian plane, let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the graphs of the quadratic functions $f_1(x) = p_1x^2 + q_1x + r_1$ and $f_2(x) = p_2x^2 + q_2x + r_2$, where $p_1 > 0 > p_2$. The graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ cross at distinct points A and B. The four tangents to $\mathcal{G}_1$ and $\mathcal{G}_2$ at A and B form a convex quadrilateral which has an inscribed circle. Prove that the graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ have the same axis of symmetry.

Problem 5. Fix an integer $n \geq 2$. An $n \times n$ *sieve* is an $n \times n$ array with n cells removed so that exactly one cell is removed from every row and every column. A *stick* is a $1 \times k$ or $k \times 1$ array for any positive integer k. For any sieve A, let $m(A)$ be the minimal number of sticks required to partition A. Find all possible values of $m(A)$, as A varies over all possible $n \times n$ sieves.

Problem 6. Let $ABCD$ be any convex quadrilateral and let P, Q, R, S be points on the segments AB, BC, CD, and DA, respectively. It is given that the segments PR and QS dissect $ABCD$ into four quadrilaterals, each of which has perpendicular diagonals. Show that the points P, Q, R, S are concyclic.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.