2nd Olympiad of Metropolises

Mathematics · Day 1

Problem 1. Let $ABCD$ be a parallelogram in which the angle at B is obtuse and $AD > AB$. Points K and L are chosen on the diagonal AC such that $\angle ABK = \angle ADL$ (the points A, K, L, C are all different, with K between A and L). The line BK intersects the circumcircle ω of triangle ABC at points B and E, and the line EL intersects ω at points E and F. Prove that $BF \parallel AC$.

Problem 2. In a country there are two-way non-stop flights between some pairs of cities. Any city can be reached from any other by a sequence of at most 100 flights. Moreover, any city can be reached from any other by a sequence of an even number of flights. What is the smallest positive integer d for which one can always claim that any city can be reached from any other one by a sequence of an even number of flights not exceeding d?

(It is allowed to visit some cities or take some flights more than once.)

Problem 3. Let $Q(t)$ be a quadratic polynomial with two distinct real zeros. Prove that there exists a non-constant polynomial $P(x)$ with the leading coefficient 1 such that the absolute values of all coefficients of the polynomial $Q(P(x))$ other than the leading one are less than 0.001.

2nd Olympiad of Metropolises

Mathematics · Day 2

Problem 4. Find the largest positive integer N for which one can choose N distinct numbers from the set $\{1, 2, 3, \ldots, 100\}$ such that neither the sum nor the product of any two different chosen numbers is divisible by 100.

Problem 5. Let x and y be positive integers greater than 1 such that

$$[x+2, y+2] - [x+1, y+1] = [x+1, y+1] - [x, y].$$

Prove that one of the two numbers x and y divides the other.

(Here $[a, b]$ denotes the least common multiple of a and b.)

Problem 6. Let $ABCDEF$ be a convex hexagon which has an inscribed circle and a circumscribed circle. Denote by ω_A, ω_B, ω_C, ω_D, ω_E, and ω_F the inscribed circles of the triangles FAB, ABC, BCD, CDE, DEF, and EFA, respectively. Let ℓ_{AB} be the external common tangent of ω_A and ω_B other than the line AB; lines ℓ_{BC}, ℓ_{CD}, ℓ_{DE}, ℓ_{EF}, and ℓ_{FA} are analogously defined. Let A_1 be the intersection point of the lines ℓ_{FA} and ℓ_{AB}; B_1 be the intersection point of the lines ℓ_{AB} and ℓ_{BC}; points C_1, D_1, E_1, and F_1 are analogously defined.

Suppose that $A_1B_1C_1D_1E_1F_1$ is a convex hexagon. Show that its diagonals A_1D_1, B_1E_1, and C_1F_1 meet at a single point.