Shortlisted Problems with Solutions

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Note of Confidentiality

The shortlisted problems should be kept strictly confidential until IMO 2017.

Contributing Countries

The Organising Committee and the Problem Selection Committee of IMO 2016 thank the following 40 countries for contributing 121 problem proposals:

Albania, Algeria, Armenia, Australia, Austria, Belarus, Belgium, Bulgaria, Colombia, Cyprus, Czech Republic, Denmark, Estonia, France, Georgia, Greece, Iceland, India, Iran, Ireland, Israel, Japan, Latvia, Luxembourg, Malaysia, Mexico, Mongolia, Netherlands, Philippines, Russia, Serbia, Slovakia, Slovenia, South Africa, Taiwan, Tanzania, Thailand, Trinidad and Tobago, Turkey, Ukraine.

Problem Selection Committee

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Problems

Algebra

A1. Let a, b and c be positive real numbers such that $\min\{ab, bc, ca\} \geqslant 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leqslant \left(\frac{a+b+c}{3}\right)^2 + 1.$$

A2. Find the smallest real constant C such that for any positive real numbers a_1, a_2, a_3, a_4 and a_5 (not necessarily distinct), one can always choose distinct subscripts i, j, k and l such that

$$\left|\frac{a_i}{a_j} - \frac{a_k}{a_l}\right| \leqslant C.$$

A3. Find all integers $n \geqslant 3$ with the following property: for all real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ satisfying $|a_k| + |b_k| = 1$ for $1 \leqslant k \leqslant n$, there exist $x_1, x_2, \ldots, x_n$, each of which is either -1 or 1, such that

$$\left|\sum_{k=1}^{n} x_k a_k\right| + \left|\sum_{k=1}^{n} x_k b_k\right| \leqslant 1.$$

A4. Denote by $\mathbb{R}^+$ the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$xf(x^2)f(f(y)) + f(yf(x)) = f(xy)\left(f(f(x^2)) + f(f(y^2))\right)$$

for all positive real numbers x and y.

A5.

(a) Prove that for every positive integer n, there exists a fraction $\frac{a}{b}$ where a and b are integers satisfying $0 < b \leqslant \sqrt{n} + 1$ and $\sqrt{n} \leqslant \frac{a}{b} \leqslant \sqrt{n+1}$.

(b) Prove that there are infinitely many positive integers n such that there is no fraction $\frac{a}{b}$ where a and b are integers satisfying $0 < b \leqslant \sqrt{n}$ and $\sqrt{n} \leqslant \frac{a}{b} \leqslant \sqrt{n+1}$.

A6. The equation

$$(x-1)(x-2)\cdots(x-2016)=(x-1)(x-2)\cdots(x-2016)$$

is written on the board. One tries to erase some linear factors from both sides so that each side still has at least one factor, and the resulting equation has no real roots. Find the least number of linear factors one needs to erase to achieve this.

A7. Denote by $\mathbb{R}$ the set of all real numbers. Find all functions $f:\mathbb{R}\rightarrow\mathbb{R}$ such that $f(0)\neq 0$ and

$$f(x+y)^2=2f(x)f(y)+\max\left\{f(x^2)+f(y^2),f(x^2+y^2)\right\}$$

for all real numbers x and y.

A8. Determine the largest real number a such that for all $n\geqslant 1$ and for all real numbers $x_0,x_1,\ldots,x_n$ satisfying $0=x_0<x_1<x_2<\cdots<x_n$, we have

$$\frac{1}{x_1-x_0}+\frac{1}{x_2-x_1}+\cdots+\frac{1}{x_n-x_{n-1}}\geqslant a\left(\frac{2}{x_1}+\frac{3}{x_2}+\cdots+\frac{n+1}{x_n}\right).$$

Combinatorics

C1. The leader of an IMO team chooses positive integers n and k with $n > k$, and announces them to the deputy leader and a contestant. The leader then secretly tells the deputy leader an n-digit binary string, and the deputy leader writes down all n-digit binary strings which differ from the leader's in exactly k positions. (For example, if $n = 3$ and $k = 1$, and if the leader chooses 101, the deputy leader would write down 001, 111 and 100.) The contestant is allowed to look at the strings written by the deputy leader and guess the leader's string. What is the minimum number of guesses (in terms of n and k) needed to guarantee the correct answer?

C2. Find all positive integers n for which all positive divisors of n can be put into the cells of a rectangular table under the following constraints:

- each cell contains a distinct divisor;
- the sums of all rows are equal; and
- the sums of all columns are equal.

C3. Let n be a positive integer relatively prime to 6. We paint the vertices of a regular n-gon with three colours so that there is an odd number of vertices of each colour. Show that there exists an isosceles triangle whose three vertices are of different colours.

C4. Find all positive integers n for which we can fill in the entries of an $n \times n$ table with the following properties:

- each entry can be one of I, M and O;
- in each row and each column, the letters I, M and O occur the same number of times; and
- in any diagonal whose number of entries is a multiple of three, the letters I, M and O occur the same number of times.

C5. Let $n \geqslant 3$ be a positive integer. Find the maximum number of diagonals of a regular n-gon one can select, so that any two of them do not intersect in the interior or they are perpendicular to each other.

C6. There are $n \geqslant 3$ islands in a city. Initially, the ferry company offers some routes between some pairs of islands so that it is impossible to divide the islands into two groups such that no two islands in different groups are connected by a ferry route.

After each year, the ferry company will close a ferry route between some two islands X and Y. At the same time, in order to maintain its service, the company will open new routes according to the following rule: for any island which is connected by a ferry route to exactly one of X and Y, a new route between this island and the other of X and Y is added.

Suppose at any moment, if we partition all islands into two nonempty groups in any way, then it is known that the ferry company will close a certain route connecting two islands from the two groups after some years. Prove that after some years there will be an island which is connected to all other islands by ferry routes.

C7. Let $n \geqslant 2$ be an integer. In the plane, there are n segments given in such a way that any two segments have an intersection point in the interior, and no three segments intersect at a single point. Jeff places a snail at one of the endpoints of each of the segments and claps his hands $n-1$ times. Each time when he claps his hands, all the snails move along their own segments and stay at the next intersection points until the next clap. Since there are $n-1$ intersection points on each segment, all snails will reach the furthest intersection points from their starting points after $n-1$ claps.

(a) Prove that if n is odd then Jeff can always place the snails so that no two of them ever occupy the same intersection point.

(b) Prove that if n is even then there must be a moment when some two snails occupy the same intersection point no matter how Jeff places the snails.

C8. Let n be a positive integer. Determine the smallest positive integer k with the following property: it is possible to mark k cells on a $2n \times 2n$ board so that there exists a unique partition of the board into 1×2 and 2×1 dominoes, none of which contains two marked cells.

Geometry

G1. In a convex pentagon $ABCDE$, let F be a point on AC such that $\angle FBC = 90°$. Suppose triangles ABF, ACD and ADE are similar isosceles triangles with

$$\angle FAB = \angle FBA = \angle DAC = \angle DCA = \angle EAD = \angle EDA.$$

Let M be the midpoint of CF. Point X is chosen such that $AMXE$ is a parallelogram. Show that BD, EM and FX are concurrent.

G2. Let ABC be a triangle with circumcircle Γ and incentre I. Let M be the midpoint of side BC. Denote by D the foot of perpendicular from I to side BC. The line through I perpendicular to AI meets sides AB and AC at F and E respectively. Suppose the circumcircle of triangle AEF intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.

G3. Let $B = (-1, 0)$ and $C = (1, 0)$ be fixed points on the coordinate plane. A nonempty, bounded subset S of the plane is said to be *nice* if

(i) there is a point T in S such that for every point Q in S, the segment TQ lies entirely in S; and

(ii) for any triangle $P_1P_2P_3$, there exists a unique point A in S and a permutation σ of the indices $\{1, 2, 3\}$ for which triangles ABC and $P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$ are similar.

Prove that there exist two distinct nice subsets S and S' of the set $\{(x, y) : x \geqslant 0, y \geqslant 0\}$ such that if $A \in S$ and $A' \in S'$ are the unique choices of points in (ii), then the product $BA \cdot BA'$ is a constant independent of the triangle $P_1P_2P_3$.

G4. Let ABC be a triangle with $AB = AC \neq BC$ and let I be its incentre. The line BI meets AC at D, and the line through D perpendicular to AC meets AI at E. Prove that the reflection of I in AC lies on the circumcircle of triangle BDE.

G5. Let D be the foot of perpendicular from A to the Euler line (the line passing through the circumcentre and the orthocentre) of an acute scalene triangle ABC. A circle ω with centre S passes through A and D, and it intersects sides AB and AC at X and Y respectively. Let P be the foot of altitude from A to BC, and let M be the midpoint of BC. Prove that the circumcentre of triangle XSY is equidistant from P and M.

G6. Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle ADC < 90^\circ$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at E and F respectively, and meet each other at point P. Let M be the midpoint of AC and let ω be the circumcircle of triangle BPD. Segments BM and DM intersect ω again at X and Y respectively. Denote by Q the intersection point of lines XE and YF. Prove that $PQ \perp AC$.

G7. Let I be the incentre of a non-equilateral triangle ABC, I_A be the A-excentre, I'_A be the reflection of I_A in BC, and l_A be the reflection of line AI'_A in AI. Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B.

(a) Prove that P lies on line OI where O is the circumcentre of triangle ABC.

(b) Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y. Show that $\angle XIY = 120^\circ$.

G8. Let A_1, B_1 and C_1 be points on sides BC, CA and AB of an acute triangle ABC respectively, such that AA_1, BB_1 and CC_1 are the internal angle bisectors of triangle ABC. Let I be the incentre of triangle ABC, and H be the orthocentre of triangle $A_1B_1C_1$. Show that

$$AH + BH + CH \geqslant AI + BI + CI.$$

Number Theory

N1. For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geqslant 2016$, the integer $P(n)$ is positive and

$$S(P(n)) = P(S(n)).$$

N2. Let $\tau(n)$ be the number of positive divisors of n. Let $\tau_1(n)$ be the number of positive divisors of n which have remainders 1 when divided by 3. Find all possible integral values of the fraction $\frac{\tau(10n)}{\tau_1(10n)}$.

N3. Define $P(n) = n^2 + n + 1$. For any positive integers a and b, the set

$$\{P(a), P(a+1), P(a+2), \ldots, P(a+b)\}$$

is said to be *fragrant* if none of its elements is relatively prime to the product of the other elements. Determine the smallest size of a fragrant set.

N4. Let n, m, k and l be positive integers with $n \neq 1$ such that $n^k + mn^l + 1$ divides $n^{k+l} - 1$. Prove that

- $m = 1$ and $l = 2k$; or
- $l | k$ and $m = \frac{n^{k-l}-1}{n^l-1}$.

N5. Let a be a positive integer which is not a square number. Denote by A the set of all positive integers k such that

$$k = \frac{x^2 - a}{x^2 - y^2} \qquad (1)$$

for some integers x and y with $x > \sqrt{a}$. Denote by B the set of all positive integers k such that (1) is satisfied for some integers x and y with $0 \leqslant x < \sqrt{a}$. Prove that $A = B$.

N6. Denote by $\mathbb{N}$ the set of all positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers m and n, the integer $f(m) + f(n) - mn$ is nonzero and divides $mf(m) + nf(n)$.

N7. Let n be an odd positive integer. In the Cartesian plane, a cyclic polygon P with area S is chosen. All its vertices have integral coordinates, and all squares of its side lengths are divisible by n. Prove that $2S$ is an integer divisible by n.

N8. Find all polynomials $P(x)$ of odd degree d and with integer coefficients satisfying the following property: for each positive integer n, there exist n positive integers $x_1, x_2, \ldots, x_n$ such that $\frac{1}{2} < \frac{P(x_i)}{P(x_j)} < 2$ and $\frac{P(x_i)}{P(x_j)}$ is the d-th power of a rational number for every pair of indices i and j with $1 \leqslant i, j \leqslant n$.

Solutions

Algebra

A1. Let a, b and c be positive real numbers such that $\min\{ab, bc, ca\} \geqslant 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leqslant \left(\frac{a+b+c}{3}\right)^2 + 1. \tag{1}$$

Solution 1. We first show the following.

- **Claim.** For any positive real numbers x, y with $xy \geqslant 1$, we have

$$(x^2+1)(y^2+1) \leqslant \left(\left(\frac{x+y}{2}\right)^2 + 1\right)^2. \tag{2}$$

Proof. Note that $xy \geqslant 1$ implies $(\frac{x+y}{2})^2 - 1 \geqslant xy - 1 \geqslant 0$. We find that

$$(x^2+1)(y^2+1) = (xy-1)^2 + (x+y)^2 \leqslant \left(\left(\frac{x+y}{2}\right)^2 - 1\right)^2 + (x+y)^2 = \left(\left(\frac{x+y}{2}\right)^2 + 1\right)^2.$$

□

Without loss of generality, assume $a \geqslant b \geqslant c$. This implies $a \geqslant 1$. Let $d = \frac{a+b+c}{3}$. Note that

$$ad = \frac{a(a+b+c)}{3} \geqslant \frac{1+1+1}{3} = 1.$$

Then we can apply (2) to the pair (a, d) and the pair (b, c). We get

$$(a^2+1)(d^2+1)(b^2+1)(c^2+1) \leqslant \left(\left(\frac{a+d}{2}\right)^2 + 1\right)^2 \left(\left(\frac{b+c}{2}\right)^2 + 1\right)^2. \tag{3}$$

Next, from

$$\frac{a+d}{2} \cdot \frac{b+c}{2} \geqslant \sqrt{ad} \cdot \sqrt{bc} \geqslant 1,$$

we can apply (2) again to the pair $(\frac{a+d}{2}, \frac{b+c}{2})$. Together with (3), we have

$$(a^2+1)(d^2+1)(b^2+1)(c^2+1) \leqslant \left(\left(\frac{a+b+c+d}{4}\right)^2 + 1\right)^4 = (d^2+1)^4.$$

Therefore, $(a^2+1)(b^2+1)(c^2+1) \leqslant (d^2+1)^3$, and (1) follows by taking cube root of both sides.

Comment. After justifying the Claim, one may also obtain (1) by mixing variables. Indeed, the function involved is clearly continuous, and hence it suffices to check that the condition $xy \geqslant 1$ is preserved under each mixing step. This is true since whenever $ab, bc, ca \geqslant 1$, we have

$$\frac{a+b}{2} \cdot \frac{a+b}{2} \geqslant ab \geqslant 1 \quad \text{and} \quad \frac{a+b}{2} \cdot c \geqslant \frac{1+1}{2} = 1.$$

Solution 2. Let $f(x) = \ln(1+x^2)$. Then the inequality (1) to be shown is equivalent to

$$\frac{f(a)+f(b)+f(c)}{3} \leqslant f\left(\frac{a+b+c}{3}\right),$$

while (2) becomes

$$\frac{f(x)+f(y)}{2} \leqslant f\left(\frac{x+y}{2}\right)$$

for $xy \geqslant 1$.

Without loss of generality, assume $a \geqslant b \geqslant c$. From the Claim in Solution 1, we have

$$\frac{f(a)+f(b)+f(c)}{3} \leqslant \frac{f(a)+2f(\frac{b+c}{2})}{3}.$$

Note that $a \geqslant 1$ and $\frac{b+c}{2} \geqslant \sqrt{bc} \geqslant 1$. Since

$$f''(x) = \frac{2(1-x^2)}{(1+x^2)^2},$$

we know that f is concave on $[1, \infty)$. Then we can apply Jensen's Theorem to get

$$\frac{f(a)+2f(\frac{b+c}{2})}{3} \leqslant f\left(\frac{a+2\cdot\frac{b+c}{2}}{3}\right) = f\left(\frac{a+b+c}{3}\right).$$

This completes the proof.

A2. Find the smallest real constant C such that for any positive real numbers a_1, a_2, a_3, a_4 and a_5 (not necessarily distinct), one can always choose distinct subscripts i, j, k and l such that

$$\left|\frac{a_i}{a_j} - \frac{a_k}{a_l}\right| \leqslant C. \tag{1}$$

__**Answer.**__ The smallest C is $\frac{1}{2}$.

__**Solution.**__ We first show that $C \leqslant \frac{1}{2}$. For any positive real numbers $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4 \leqslant a_5$, consider the five fractions

$$\frac{a_1}{a_2}, \frac{a_3}{a_4}, \frac{a_1}{a_5}, \frac{a_2}{a_3}, \frac{a_4}{a_5}. \tag{2}$$

Each of them lies in the interval $(0, 1]$. Therefore, by the Pigeonhole Principle, at least three of them must lie in $(0, \frac{1}{2}]$ or lie in $(\frac{1}{2}, 1]$ simultaneously. In particular, there must be two consecutive terms in (2) which belong to an interval of length $\frac{1}{2}$ (here, we regard $\frac{a_1}{a_2}$ and $\frac{a_4}{a_5}$ as consecutive). In other words, the difference of these two fractions is less than $\frac{1}{2}$. As the indices involved in these two fractions are distinct, we can choose them to be i, j, k, l and conclude that $C \leqslant \frac{1}{2}$.

Next, we show that $C = \frac{1}{2}$ is best possible. Consider the numbers $1, 2, 2, 2, n$ where n is a large real number. The fractions formed by two of these numbers in ascending order are $\frac{1}{n}, \frac{2}{n}, \frac{1}{2}, \frac{2}{2}, \frac{2}{1}, \frac{n}{2}, \frac{n}{1}$. Since the indices i, j, k, l are distinct, $\frac{1}{n}$ and $\frac{2}{n}$ cannot be chosen simultaneously. Therefore the minimum value of the left-hand side of (1) is $\frac{1}{2} - \frac{2}{n}$. When n tends to infinity, this value approaches $\frac{1}{2}$, and so C cannot be less than $\frac{1}{2}$.

These conclude that $C = \frac{1}{2}$ is the smallest possible choice.

__**Comment.**__ The conclusion still holds if $a_1, a_2, \ldots, a_5$ are pairwise distinct, since in the construction, we may replace the 2's by real numbers sufficiently close to 2.

There are two possible simplifications for this problem:

(i) the answer $C = \frac{1}{2}$ is given to the contestants; or

(ii) simply ask the contestants to prove the inequality (1) for $C = \frac{1}{2}$.

A3. Find all integers $n \geqslant 3$ with the following property: for all real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ satisfying $|a_k| + |b_k| = 1$ for $1 \leqslant k \leqslant n$, there exist $x_1, x_2, \ldots, x_n$, each of which is either -1 or 1, such that

$$\left|\sum_{k=1}^{n} x_k a_k\right| + \left|\sum_{k=1}^{n} x_k b_k\right| \leqslant 1. \tag{1}$$

Answer. n can be any odd integer greater than or equal to 3.

Solution 1. For any even integer $n \geqslant 4$, we consider the case

$$a_1 = a_2 = \cdots = a_{n-1} = b_n = 0 \quad \text{and} \quad b_1 = b_2 = \cdots = b_{n-1} = a_n = 1.$$

The condition $|a_k| + |b_k| = 1$ is satisfied for each $1 \leqslant k \leqslant n$. No matter how we choose each x_k, both sums $\sum_{k=1}^{n} x_k a_k$ and $\sum_{k=1}^{n} x_k b_k$ are odd integers. This implies $|\sum_{k=1}^{n} x_k a_k| \geqslant 1$ and $|\sum_{k=1}^{n} x_k b_k| \geqslant 1$, which shows (1) cannot hold.

For any odd integer $n \geqslant 3$, we may assume without loss of generality $b_k \geqslant 0$ for $1 \leqslant k \leqslant n$ (this can be done by flipping the pair (a_k, b_k) to $(-a_k, -b_k)$ and x_k to $-x_k$ if necessary) and $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_m \geqslant 0 > a_{m+1} \geqslant \cdots \geqslant a_n$. We claim that the choice $x_k = (-1)^{k+1}$ for $1 \leqslant k \leqslant n$ will work. Define

$$s = \sum_{k=1}^{m} x_k a_k \quad \text{and} \quad t = -\sum_{k=m+1}^{n} x_k a_k.$$

Note that

$$s = (a_1 - a_2) + (a_3 - a_4) + \cdots \geqslant 0$$

by the assumption $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_m$ (when m is odd, there is a single term a_m at the end, which is also positive). Next, we have

$$s = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots \leqslant a_1 \leqslant 1.$$

Similarly,

$$t = (-a_n + a_{n-1}) + (-a_{n-2} + a_{n-3}) + \cdots \geqslant 0$$

and

$$t = -a_n + (a_{n-1} - a_{n-2}) + (a_{n-3} - a_{n-4}) + \cdots \leqslant -a_n \leqslant 1.$$

From the condition, we have $a_k + b_k = 1$ for $1 \leqslant k \leqslant m$ and $-a_k + b_k = 1$ for $m+1 \leqslant k \leqslant n$. It follows that $\sum_{k=1}^{n} x_k a_k = s - t$ and $\sum_{k=1}^{n} x_k b_k = 1 - s - t$. Hence it remains to prove

$$|s - t| + |1 - s - t| \leqslant 1$$

under the constraint $0 \leqslant s, t \leqslant 1$. By symmetry, we may assume $s \geqslant t$. If $1 - s - t \geqslant 0$, then we have

$$|s - t| + |1 - s - t| = s - t + 1 - s - t = 1 - 2t \leqslant 1.$$

If $1 - s - t \leqslant 0$, then we have

$$|s - t| + |1 - s - t| = s - t - 1 + s + t = 2s - 1 \leqslant 1.$$

Hence, the inequality is true in both cases.

These show n can be any odd integer greater than or equal to 3.

Solution 2. The even case can be handled in the same way as Solution 1. For the odd case, we prove by induction on n.

Firstly, for $n = 3$, we may assume without loss of generality $a_1 \geqslant a_2 \geqslant a_3 \geqslant 0$ and $b_1 = a_1 - 1$ (if $b_1 = 1 - a_1$, we may replace each b_k by $-b_k$).

- **Case 1.** $b_2 = a_2 - 1$ and $b_3 = a_3 - 1$, in which case we take $(x_1, x_2, x_3) = (1, -1, 1)$.

  Let $c = a_1 - a_2 + a_3$ so that $0 \leqslant c \leqslant 1$. Then $|b_1 - b_2 + b_3| = |a_1 - a_2 + a_3 - 1| = 1 - c$ and hence $|c| + |b_1 - b_2 + b_3| = 1$.

- **Case 2.** $b_2 = 1 - a_2$ and $b_3 = 1 - a_3$, in which case we take $(x_1, x_2, x_3) = (1, -1, 1)$.

  Let $c = a_1 - a_2 + a_3$ so that $0 \leqslant c \leqslant 1$. Since $a_3 \leqslant a_2$ and $a_1 \leqslant 1$, we have

$$c - 1 \leqslant b_1 - b_2 + b_3 = a_1 + a_2 - a_3 - 1 \leqslant 1 - c.$$

This gives $|b_1 - b_2 + b_3| \leqslant 1 - c$ and hence $|c| + |b_1 - b_2 + b_3| \leqslant 1$.

- **Case 3.** $b_2 = a_2 - 1$ and $b_3 = 1 - a_3$, in which case we take $(x_1, x_2, x_3) = (-1, 1, 1)$.

  Let $c = -a_1 + a_2 + a_3$. If $c \geqslant 0$, then $a_3 \leqslant 1$ and $a_2 \leqslant a_1$ imply

$$c - 1 \leqslant -b_1 + b_2 + b_3 = -a_1 + a_2 - a_3 + 1 \leqslant 1 - c.$$

  If $c < 0$, then $a_1 \leqslant a_2 + 1$ and $a_3 \geqslant 0$ imply

$$-c - 1 \leqslant -b_1 + b_2 + b_3 = -a_1 + a_2 - a_3 + 1 \leqslant 1 + c.$$

  In both cases, we get $|-b_1 + b_2 + b_3| \leqslant 1 - |c|$ and hence $|c| + |-b_1 + b_2 + b_3| \leqslant 1$.

- **Case 4.** $b_2 = 1 - a_2$ and $b_3 = a_3 - 1$, in which case we take $(x_1, x_2, x_3) = (-1, 1, 1)$.

  Let $c = -a_1 + a_2 + a_3$. If $c \geqslant 0$, then $a_2 \leqslant 1$ and $a_3 \leqslant a_1$ imply

$$c - 1 \leqslant -b_1 + b_2 + b_3 = -a_1 - a_2 + a_3 + 1 \leqslant 1 - c.$$

  If $c < 0$, then $a_1 \leqslant a_3 + 1$ and $a_2 \geqslant 0$ imply

$$-c - 1 \leqslant -b_1 + b_2 + b_3 = -a_1 - a_2 + a_3 + 1 \leqslant 1 + c.$$

  In both cases, we get $|-b_1 + b_2 + b_3| \leqslant 1 - |c|$ and hence $|c| + |-b_1 + b_2 + b_3| \leqslant 1$.

We have found x_1, x_2, x_3 satisfying (1) in each case for $n = 3$.

Now, let $n \geqslant 5$ be odd and suppose the result holds for any smaller odd cases. Again we may assume $a_k \geqslant 0$ for each $1 \leqslant k \leqslant n$. By the Pigeonhole Principle, there are at least three indices k for which $b_k = a_k - 1$ or $b_k = 1 - a_k$. Without loss of generality, suppose $b_k = a_k - 1$ for $k = 1, 2, 3$. Again by the Pigeonhole Principle, as a_1, a_2, a_3 lies between 0 and 1, the difference of two of them is at most $\frac{1}{2}$. By changing indices if necessary, we may assume $0 \leqslant d = a_1 - a_2 \leqslant \frac{1}{2}$.

By the inductive hypothesis, we can choose $x_3, x_4, \ldots, x_n$ such that $a' = \sum_{k=3}^{n} x_k a_k$ and $b' = \sum_{k=3}^{n} x_k b_k$ satisfy $|a'| + |b'| \leqslant 1$. We may further assume $a' \geqslant 0$.

- **Case 1.** $b' \geqslant 0$, in which case we take $(x_1, x_2) = (-1, 1)$.

  We have $|-a_1 + a_2 + a'| + |-(a_1 - 1) + (a_2 - 1) + b'| = |-d + a'| + |-d + b'| \leqslant \max\{a' + b' - 2d, a' - b', b' - a', 2d - a' - b'\} \leqslant 1$ since $0 \leqslant a', b', a' + b' \leqslant 1$ and $0 \leqslant d \leqslant \frac{1}{2}$.

- **Case 2.** $0 > b' \geqslant -a'$, in which case we take $(x_1, x_2) = (-1, 1)$.

  We have $|-a_1 + a_2 + a'| + |-(a_1 - 1) + (a_2 - 1) + b'| = |-d + a'| + |-d + b'|$. If $-d + a' \geqslant 0$, this equals $a' - b' = |a'| + |b'| \leqslant 1$. If $-d + a' < 0$, this equals $2d - a' - b' \leqslant 2d \leqslant 1$.

- **Case 3.** $b' < -a'$, in which case we take $(x_1, x_2) = (1, -1)$.

  We have $|a_1 - a_2 + a'| + |(a_1 - 1) - (a_2 - 1) + b'| = |d + a'| + |d + b'|$. If $d + b' \geqslant 0$, this equals $2d + a' + b' < 2d \leqslant 1$. If $d + b' < 0$, this equals $a' - b' = |a'| + |b'| \leqslant 1$.

Therefore, we have found $x_1, x_2, \ldots, x_n$ satisfying (1) in each case. By induction, the property holds for all odd integers $n \geqslant 3$.

A4. Denote by $\mathbb{R}^+$ the set of all positive real numbers. Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$xf(x^2)f(f(y))+f(yf(x))=f(xy)\left(f(f(x^2))+f(f(y^2))\right) \tag{1}$$

for all positive real numbers x and y.

Answer. $f(x)=\frac{1}{x}$ for any $x\in\mathbb{R}^+$.

Solution 1. Taking $x=y=1$ in (1), we get $f(1)f(f(1))+f(f(1))=2f(1)f(f(1))$ and hence $f(1)=1$. Swapping x and y in (1) and comparing with (1) again, we find

$$xf(x^2)f(f(y))+f(yf(x))=yf(y^2)f(f(x))+f(xf(y)). \tag{2}$$

Taking $y=1$ in (2), we have $xf(x^2)+f(f(x))=f(f(x))+f(x)$, that is,

$$f(x^2)=\frac{f(x)}{x}. \tag{3}$$

Take $y=1$ in (1) and apply (3) to $xf(x^2)$. We get $f(x)+f(f(x))=f(x)(f(f(x^2))+1)$, which implies

$$f(f(x^2))=\frac{f(f(x))}{f(x)}. \tag{4}$$

For any $x\in\mathbb{R}^+$, we find that

$$f(f(x)^2)\overset{(3)}{=}\frac{f(f(x))}{f(x)}\overset{(4)}{=}f(f(x^2))\overset{(3)}{=}f\left(\frac{f(x)}{x}\right). \tag{5}$$

It remains to show the following key step.

• **Claim.** The function f is injective.

Proof. Using (3) and (4), we rewrite (1) as

$$f(x)f(f(y))+f(yf(x))=f(xy)\left(\frac{f(f(x))}{f(x)}+\frac{f(f(y))}{f(y)}\right). \tag{6}$$

Take $x=y$ in (6) and apply (3). This gives $f(x)f(f(x))+f(xf(x))=2\frac{f(f(x))}{x}$, which means

$$f(xf(x))=f(f(x))\left(\frac{2}{x}-f(x)\right). \tag{7}$$

Using (3), equation (2) can be rewritten as

$$f(x)f(f(y))+f(yf(x))=f(y)f(f(x))+f(xf(y)). \tag{8}$$

Suppose $f(x)=f(y)$ for some $x,y\in\mathbb{R}^+$. Then (8) implies

$$f(yf(y))=f(yf(x))=f(xf(y))=f(xf(x)).$$

Using (7), this gives

$$f(f(y))\left(\frac{2}{y}-f(y)\right)=f(f(x))\left(\frac{2}{x}-f(x)\right).$$

Noting $f(x)=f(y)$, we find $x=y$. This establishes the injectivity. □

By the Claim and (5), we get the only possible solution $f(x)=\frac{1}{x}$. It suffices to check that this is a solution. Indeed, the left-hand side of (1) becomes

$$x\cdot\frac{1}{x^2}\cdot y+\frac{x}{y}=\frac{y}{x}+\frac{x}{y},$$

while the right-hand side becomes

$$\frac{1}{xy}(x^2+y^2)=\frac{x}{y}+\frac{y}{x}.$$

The two sides agree with each other.

Solution 2. Taking $x=y=1$ in (1), we get $f(1)f(f(1))+f(f(1))=2f(1)f(f(1))$ and hence $f(1)=1$. Putting $x=1$ in (1), we have $f(f(y))+f(y)=f(y)(1+f(f(y^2)))$ so that

$$f(f(y))=f(y)f(f(y^2)). \tag{9}$$

Putting $y=1$ in (1), we get $xf(x^2)+f(f(x))=f(x)(f(f(x^2))+1)$. Using (9), this gives

$$xf(x^2)=f(x). \tag{10}$$

Replace y by $\frac{1}{x}$ in (1). Then we have

$$xf(x^2)f\left(f\left(\frac{1}{x}\right)\right)+f\left(\frac{f(x)}{x}\right)=f(f(x^2))+f\left(f\left(\frac{1}{x^2}\right)\right).$$

The relation (10) shows $f(\frac{f(x)}{x})=f(f(x^2))$. Also, using (9) with $y=\frac{1}{x}$ and using (10) again, the last equation reduces to

$$f(x)f\left(\frac{1}{x}\right)=1. \tag{11}$$

Replace x by $\frac{1}{x}$ and y by $\frac{1}{y}$ in (1) and apply (11). We get

$$\frac{1}{xf(x^2)f(f(y))}+\frac{1}{f(yf(x))}=\frac{1}{f(xy)}\left(\frac{1}{f(f(x^2))}+\frac{1}{f(f(y^2))}\right).$$

Clearing denominators, we can use (1) to simplify the numerators and obtain

$$f(xy)^2f(f(x^2))f(f(y^2))=xf(x^2)f(f(y))f(yf(x)).$$

Using (9) and (10), this is the same as

$$f(xy)^2f(f(x))=f(x)^2f(y)f(yf(x)). \tag{12}$$

Substitute $y=f(x)$ in (12) and apply (10) (with x replaced by $f(x)$). We have

$$f(xf(x))^2=f(x)f(f(x)). \tag{13}$$

Taking $y=x$ in (12), squaring both sides, and using (10) and (13), we find that

$$f(f(x))=x^4f(x)^3. \tag{14}$$

Finally, we combine (9), (10) and (14) to get

$$y^4f(y)^3\overset{(14)}{=}f(f(y))\overset{(9)}{=}f(y)f(f(y^2))\overset{(14)}{=}f(y)y^8f(y^2)^3\overset{(10)}{=}y^5f(y)^4,$$

which implies $f(y)=\frac{1}{y}$. This is a solution by the checking in Solution 1.

A5.

(a) Prove that for every positive integer n, there exists a fraction $\frac{a}{b}$ where a and b are integers satisfying $0 < b \leqslant \sqrt{n} + 1$ and $\sqrt{n} \leqslant \frac{a}{b} \leqslant \sqrt{n+1}$.

(b) Prove that there are infinitely many positive integers n such that there is no fraction $\frac{a}{b}$ where a and b are integers satisfying $0 < b \leqslant \sqrt{n}$ and $\sqrt{n} \leqslant \frac{a}{b} \leqslant \sqrt{n+1}$.

Solution.

(a) Let r be the unique positive integer for which $r^2 \leqslant n < (r+1)^2$. Write $n = r^2 + s$. Then we have $0 \leqslant s \leqslant 2r$. We discuss in two cases according to the parity of s.

- **Case 1.** s is even.

  Consider the number $(r + \frac{s}{2r})^2 = r^2 + s + (\frac{s}{2r})^2$. We find that

$$n = r^2 + s \leqslant r^2 + s + \left(\frac{s}{2r}\right)^2 \leqslant r^2 + s + 1 = n + 1.$$

  It follows that

$$\sqrt{n} \leqslant r + \frac{s}{2r} \leqslant \sqrt{n+1}.$$

  Since s is even, we can choose the fraction $r + \frac{s}{2r} = \frac{r^2 + (s/2)}{r}$ since $r \leqslant \sqrt{n}$.

- **Case 2.** s is odd.

  Consider the number $(r + 1 - \frac{2r+1-s}{2(r+1)})^2 = (r+1)^2 - (2r+1-s) + (\frac{2r+1-s}{2(r+1)})^2$. We find that

$$\begin{aligned} n = r^2 + s = (r+1)^2 - (2r+1-s) &\leqslant (r+1)^2 - (2r+1-s) + \left(\frac{2r+1-s}{2(r+1)}\right)^2 \\ &\leqslant (r+1)^2 - (2r+1-s) + 1 = n + 1. \end{aligned}$$

  It follows that

$$\sqrt{n} \leqslant r + 1 - \frac{2r+1-s}{2(r+1)} \leqslant \sqrt{n+1}.$$

  Since s is odd, we can choose the fraction $(r+1) - \frac{2r+1-s}{2(r+1)} = \frac{(r+1)^2 - r + ((s-1)/2)}{r+1}$ since $r + 1 \leqslant \sqrt{n} + 1$.

(b) We show that for every positive integer r, there is no fraction $\frac{a}{b}$ with $b \leqslant \sqrt{r^2+1}$ such that $\sqrt{r^2+1} \leqslant \frac{a}{b} \leqslant \sqrt{r^2+2}$. Suppose on the contrary that such a fraction exists. Since $b \leqslant \sqrt{r^2+1} < r + 1$ and b is an integer, we have $b \leqslant r$. Hence,

$$(br)^2 < b^2(r^2+1) \leqslant a^2 \leqslant b^2(r^2+2) \leqslant b^2r^2 + 2br < (br+1)^2.$$

This shows the square number a^2 is strictly bounded between the two consecutive squares $(br)^2$ and $(br+1)^2$, which is impossible. Hence, we have found infinitely many $n = r^2 + 1$ for which there is no fraction of the desired form.

A6. The equation

$$(x-1)(x-2)\cdots(x-2016)=(x-1)(x-2)\cdots(x-2016)$$

is written on the board. One tries to erase some linear factors from both sides so that each side still has at least one factor, and the resulting equation has no real roots. Find the least number of linear factors one needs to erase to achieve this.

Answer. 2016.

Solution. Since there are 2016 common linear factors on both sides, we need to erase at least 2016 factors. We claim that the equation has no real roots if we erase all factors $(x-k)$ on the left-hand side with $k \equiv 2,3 \pmod 4$, and all factors $(x-m)$ on the right-hand side with $m \equiv 0,1 \pmod 4$. Therefore, it suffices to show that no real number x satisfies

$$\prod_{j=0}^{503}(x-4j-1)(x-4j-4)=\prod_{j=0}^{503}(x-4j-2)(x-4j-3). \tag{1}$$

- **Case 1.** $x=1,2,\ldots,2016$.
  In this case, one side of (1) is zero while the other side is not. This shows x cannot satisfy (1).
- **Case 2.** $4k+1<x<4k+2$ or $4k+3<x<4k+4$ for some $k=0,1,\ldots,503$.
  For $j=0,1,\ldots,503$ with $j\neq k$, the product $(x-4j-1)(x-4j-4)$ is positive. For $j=k$, the product $(x-4k-1)(x-4k-4)$ is negative. This shows the left-hand side of (1) is negative. On the other hand, each product $(x-4j-2)(x-4j-3)$ on the right-hand side of (1) is positive. This yields a contradiction.
- **Case 3.** $x<1$ or $x>2016$ or $4k<x<4k+1$ for some $k=1,2,\ldots,503$.
  The equation (1) can be rewritten as

$$1=\prod_{j=0}^{503}\frac{(x-4j-1)(x-4j-4)}{(x-4j-2)(x-4j-3)}=\prod_{j=0}^{503}\left(1-\frac{2}{(x-4j-2)(x-4j-3)}\right).$$

Note that $(x-4j-2)(x-4j-3)>2$ for $0\leqslant j\leqslant 503$ in this case. So each term in the product lies strictly between 0 and 1, and the whole product must be less than 1, which is impossible.

- **Case 4.** $4k+2<x<4k+3$ for some $k=0,1,\ldots,503$.
  This time we rewrite (1) as

$$\begin{aligned}1&=\frac{x-1}{x-2}\cdot\frac{x-2016}{x-2015}\prod_{j=1}^{503}\frac{(x-4j)(x-4j-1)}{(x-4j+1)(x-4j-2)}\\&=\frac{x-1}{x-2}\cdot\frac{x-2016}{x-2015}\prod_{j=1}^{503}\left(1+\frac{2}{(x-4j+1)(x-4j-2)}\right).\end{aligned}$$

Clearly, $\frac{x-1}{x-2}$ and $\frac{x-2016}{x-2015}$ are both greater than 1. For the range of x in this case, each term in the product is also greater than 1. Then the right-hand side must be greater than 1 and hence a contradiction arises.

From the four cases, we conclude that (1) has no real roots. Hence, the minimum number of linear factors to be erased is 2016.

Comment. We discuss the general case when 2016 is replaced by a positive integer n. The above solution works equally well when n is divisible by 4.

If $n \equiv 2 \pmod 4$, one may leave $l(x) = (x-1)(x-2)\cdots(x-\frac{n}{2})$ on the left-hand side and $r(x) = (x-\frac{n}{2}-1)(x-\frac{n}{2}-2)\cdots(x-n)$ on the right-hand side. One checks that for $x < \frac{n+1}{2}$, we have $|l(x)| < |r(x)|$, while for $x > \frac{n+1}{2}$, we have $|l(x)| > |r(x)|$.

If $n \equiv 3 \pmod 4$, one may leave $l(x) = (x-1)(x-2)\cdots(x-\frac{n+1}{2})$ on the left-hand side and $r(x) = (x-\frac{n+3}{2})(x-\frac{x+5}{2})\cdots(x-n)$ on the right-hand side. For $x < 1$ or $\frac{n+1}{2} < x < \frac{n+3}{2}$, we have $l(x) > 0 > r(x)$. For $1 < x < \frac{n+1}{2}$, we have $|l(x)| < |r(x)|$. For $x > \frac{n+3}{2}$, we have $|l(x)| > |r(x)|$.

If $n \equiv 1 \pmod 4$, as the proposer mentioned, the situation is a bit more out of control. Since the construction for $n-1 \equiv 0 \pmod 4$ works, the answer can be either n or $n-1$. For $n = 5$, we can leave the products $(x-1)(x-2)(x-3)(x-4)$ and $(x-5)$. For $n = 9$, the only example that works is $l(x) = (x-1)(x-2)(x-9)$ and $r(x) = (x-3)(x-4)\cdots(x-8)$, while there seems to be no such partition for $n = 13$.

A7. Denote by $\mathbb{R}$ the set of all real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) \neq 0$ and

$$f(x+y)^2 = 2f(x)f(y) + \max\left\{f(x^2)+f(y^2), f(x^2+y^2)\right\} \tag{1}$$

for all real numbers x and y.

Answer.

- $f(x) = -1$ for any $x \in \mathbb{R}$; or
- $f(x) = x - 1$ for any $x \in \mathbb{R}$.

Solution 1. Taking $x = y = 0$ in (1), we get $f(0)^2 = 2f(0)^2 + \max\{2f(0), f(0)\}$. If $f(0) > 0$, then $f(0)^2 + 2f(0) = 0$ gives no positive solution. If $f(0) < 0$, then $f(0)^2 + f(0) = 0$ gives $f(0) = -1$. Putting $y = 0$ in (1), we have $f(x)^2 = -2f(x) + f(x^2)$, which is the same as $(f(x)+1)^2 = f(x^2)+1$. Let $g(x) = f(x)+1$. Then for any $x \in \mathbb{R}$, we have

$$g(x^2) = g(x)^2 \geqslant 0. \tag{2}$$

From (1), we find that $f(x+y)^2 \geqslant 2f(x)f(y) + f(x^2) + f(y^2)$. In terms of g, this becomes $(g(x+y)-1)^2 \geqslant 2(g(x)-1)(g(y)-1) + g(x^2) + g(y^2) - 2$. Using (2), this means

$$(g(x+y)-1)^2 \geqslant (g(x)+g(y)-1)^2 - 1. \tag{3}$$

Putting $x = 1$ in (2), we get $g(1) = 0$ or 1. The two cases are handled separately.

- **Case 1.** $g(1) = 0$, which is the same as $f(1) = -1$.

We put $x = -1$ and $y = 0$ in (1). This gives $f(-1)^2 = -2f(-1) - 1$, which forces $f(-1) = -1$. Next, we take $x = -1$ and $y = 1$ in (1) to get $1 = 2 + \max\{-2, f(2)\}$. This clearly implies $1 = 2 + f(2)$ and hence $f(2) = -1$, that is, $g(2) = 0$. From (2), we can prove inductively that $g(2^{2^n}) = g(2)^{2^n} = 0$ for any $n \in \mathbb{N}$. Substitute $y = 2^{2^n} - x$ in (3). We obtain

$$(g(x) + g(2^{2^n} - x) - 1)^2 \leqslant (g(2^{2^n}) - 1)^2 + 1 = 2.$$

For any fixed $x \geqslant 0$, we consider n to be sufficiently large so that $2^{2^n} - x > 0$. From (2), this implies $g(2^{2^n} - x) \geqslant 0$ so that $g(x) \leqslant 1 + \sqrt{2}$. Using (2) again, we get

$$g(x)^{2^n} = g(x^{2^n}) \leqslant 1 + \sqrt{2}$$

for any $n \in \mathbb{N}$. Therefore, $|g(x)| \leqslant 1$ for any $x \geqslant 0$.

If there exists $a \in \mathbb{R}$ for which $g(a) \neq 0$, then for sufficiently large n we must have $g((a^2)^{\frac{1}{2^n}}) = g(a^2)^{\frac{1}{2^n}} > \frac{1}{2}$. By taking $x = -y = -(a^2)^{\frac{1}{2^n}}$ in (1), we obtain

$$\begin{aligned}
1 &= 2f(x)f(-x) + \max\left\{2f(x^2), f(2x^2)\right\} \\
&= 2(g(x)-1)(g(-x)-1) + \max\left\{2(g(x^2)-1), g(2x^2)-1\right\} \\
&\leqslant 2\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) + 0 = \frac{1}{2}
\end{aligned}$$

since $|g(-x)| = |g(x)| \in (\frac{1}{2}, 1]$ by (2) and the choice of x, and since $g(z) \leqslant 1$ for $z \geqslant 0$. This yields a contradiction and hence $g(x) = 0$ must hold for any x. This means $f(x) = -1$ for any $x \in \mathbb{R}$, which clearly satisfies (1).

- **Case 2.** $g(1) = 1$, which is the same as $f(1) = 0$.

We put $x = -1$ and $y = 1$ in (1) to get $1 = \max\{0, f(2)\}$. This clearly implies $f(2) = 1$ and hence $g(2) = 2$. Setting $x = 2n$ and $y = 2$ in (3), we have

$$(g(2n+2) - 1)^2 \geqslant (g(2n) + 1)^2 - 1.$$

By induction on n, it is easy to prove that $g(2n) \geqslant n + 1$ for all $n \in \mathbb{N}$. For any real number $a > 1$, we choose a large $n \in \mathbb{N}$ and take k to be the positive integer such that $2k \leqslant a^{2^n} < 2k + 2$. From (2) and (3), we have

$$(g(a)^{2^n} - 1)^2 + 1 = (g(a^{2^n}) - 1)^2 + 1 \geqslant (g(2k) + g(a^{2^n} - 2k) - 1)^2 \geqslant k^2 > \frac{1}{4}(a^{2^n} - 2)^2$$

since $g(a^{2^n} - 2k) \geqslant 0$. For large n, this clearly implies $g(a)^{2^n} > 1$. Thus,

$$(g(a)^{2^n})^2 > (g(a)^{2^n} - 1)^2 + 1 > \frac{1}{4}(a^{2^n} - 2)^2.$$

This yields

$$g(a)^{2^n} > \frac{1}{2}(a^{2^n} - 2). \tag{4}$$

Note that

$$\frac{a^{2^n}}{a^{2^n} - 2} = 1 + \frac{2}{a^{2^n} - 2} \leqslant \left(1 + \frac{2}{2^n(a^{2^n} - 2)}\right)^{2^n}$$

by binomial expansion. This can be rewritten as

$$(a^{2^n} - 2)^{\frac{1}{2^n}} \geqslant \frac{a}{1 + \frac{2}{2^n(a^{2^n} - 2)}}.$$

Together with (4), we conclude $g(a) \geqslant a$ by taking n sufficiently large.

Consider $x = na$ and $y = a > 1$ in (3). This gives $(g((n+1)a) - 1)^2 \geqslant (g(na) + g(a) - 1)^2 - 1$. By induction on n, it is easy to show $g(na) \geqslant (n-1)(g(a) - 1) + a$ for any $n \in \mathbb{N}$. We choose a large $n \in \mathbb{N}$ and take k to be the positive integer such that $ka \leqslant 2^{2^n} < (k+1)a$. Using (2) and (3), we have

$$2^{2^{n+1}} > (2^{2^n} - 1)^2 + 1 = (g(2^{2^n}) - 1)^2 + 1 \geqslant (g(2^{2^n} - ka) + g(ka) - 1)^2 \geqslant ((k-1)(g(a) - 1) + a - 1)^2,$$

from which it follows that

$$2^{2^n} \geqslant (k-1)(g(a) - 1) + a - 1 > \frac{2^{2^n}}{a}(g(a) - 1) - 2(g(a) - 1) + a - 1$$

holds for sufficiently large n. Hence, we must have $\frac{g(a)-1}{a} \leqslant 1$, which implies $g(a) \leqslant a + 1$ for any $a > 1$. Then for large $n \in \mathbb{N}$, from (3) and (2) we have

$$4a^{2^{n+1}} = (2a^{2^n})^2 \geqslant (g(2a^{2^n}) - 1)^2 \geqslant (2g(a^{2^n}) - 1)^2 - 1 = (2g(a)^{2^n} - 1)^2 - 1.$$

This implies

$$2a^{2^n} > \frac{1}{2}(1 + \sqrt{4a^{2^{n+1}} + 1}) \geqslant g(a)^{2^n}.$$

When n tends to infinity, this forces $g(a) \leqslant a$. Together with $g(a) \geqslant a$, we get $g(a) = a$ for all real numbers $a > 1$, that is, $f(a) = a - 1$ for all $a > 1$.

Finally, for any $x \in \mathbb{R}$, we choose y sufficiently large in (1) so that $y, x+y > 1$. This gives $(x+y-1)^2 = 2f(x)(y-1) + \max\{f(x^2) + y^2 - 1, x^2 + y^2 - 1\}$, which can be rewritten as

$$2(x - 1 - f(x))y = -x^2 + 2x - 2 - 2f(x) + \max\{f(x^2), x^2\}.$$

As the right-hand side is fixed, this can only hold for all large y when $f(x) = x - 1$. We now check that this function satisfies (1). Indeed, we have

$$\begin{aligned} f(x+y)^2 = (x+y-1)^2 &= 2(x-1)(y-1) + (x^2 + y^2 - 1) \\ &= 2f(x)f(y) + \max\{f(x^2) + f(y^2), f(x^2 + y^2)\}. \end{aligned}$$

Solution 2. Taking $x = y = 0$ in (1), we get $f(0)^2 = 2f(0)^2 + \max\{2f(0), f(0)\}$. If $f(0) > 0$, then $f(0)^2 + 2f(0) = 0$ gives no positive solution. If $f(0) < 0$, then $f(0)^2 + f(0) = 0$ gives $f(0) = -1$. Putting $y = 0$ in (1), we have

$$f(x)^2 = -2f(x) + f(x^2). \tag{5}$$

Replace x by $-x$ in (5) and compare with (5) again. We get $f(x)^2 + 2f(x) = f(-x)^2 + 2f(-x)$, which implies

$$f(x) = f(-x) \quad \text{or} \quad f(x) + f(-x) = -2. \tag{6}$$

Taking $x = y$ and $x = -y$ respectively in (1) and comparing the two equations obtained, we have

$$f(2x)^2 - 2f(x)^2 = 1 - 2f(x)f(-x). \tag{7}$$

Combining (6) and (7) to eliminate $f(-x)$, we find that $f(2x)$ can be ± 1 (when $f(x) = f(-x)$) or $\pm(2f(x) + 1)$ (when $f(x) + f(-x) = -2$).

We prove the following.

- **Claim.** $f(x) + f(-x) = -2$ for any $x \in \mathbb{R}$.

Proof. Suppose there exists $a \in \mathbb{R}$ such that $f(a) + f(-a) \neq -2$. Then $f(a) = f(-a) \neq -1$ and we may assume $a > 0$. We first show that $f(a) \neq 1$. Suppose $f(a) = 1$. Consider $y = a$ in (7). We get $f(2a)^2 = 1$. Taking $x = y = a$ in (1), we have $1 = 2 + \max\{2f(a^2), f(2a^2)\}$. From (5), $f(a^2) = 3$ so that $1 \geqslant 2 + 6$. This is impossible, and thus $f(a) \neq 1$.

As $f(a) \neq \pm 1$, we have $f(a) = \pm(2f(\frac{a}{2}) + 1)$. Similarly, $f(-a) = \pm(2f(-\frac{a}{2}) + 1)$. These two expressions are equal since $f(a) = f(-a)$. If $f(\frac{a}{2}) = f(-\frac{a}{2})$, then the above argument works when we replace a by $\frac{a}{2}$. In particular, we have $f(a)^2 = f(2 \cdot \frac{a}{2})^2 = 1$, which is a contradiction. Therefore, (6) forces $f(\frac{a}{2}) + f(-\frac{a}{2}) = -2$. Then we get

$$\pm\left(2f\left(\frac{a}{2}\right) + 1\right) = \pm\left(-2f\left(\frac{a}{2}\right) - 3\right).$$

For any choices of the two signs, we either get a contradiction or $f(\frac{a}{2}) = -1$, in which case $f(\frac{a}{2}) = f(-\frac{a}{2})$ and hence $f(a) = \pm 1$ again. Therefore, there is no such real number a and the Claim follows. □

Replace x and y by $-x$ and $-y$ in (1) respectively and compare with (1). We get

$$f(x+y)^2 - 2f(x)f(y) = f(-x-y)^2 - 2f(-x)f(-y).$$

Using the Claim, this simplifies to $f(x+y) = f(x)+f(y)+1$. In addition, (5) can be rewritten as $(f(x)+1)^2 = f(x^2)+1$. Therefore, the function g defined by $g(x) = f(x)+1$ satisfies $g(x+y) = g(x)+g(y)$ and $g(x)^2 = g(x^2)$. The latter relation shows $g(y)$ is nonnegative for $y \geqslant 0$. For such a function satisfying the Cauchy Equation $g(x+y) = g(x)+g(y)$, it must be monotonic increasing and hence $g(x) = cx$ for some constant c.

From $(cx)^2 = g(x)^2 = g(x^2) = cx^2$, we get $c = 0$ or 1, which corresponds to the two functions $f(x) = -1$ and $f(x) = x-1$ respectively, both of which are solutions to (1) as checked in Solution 1.

Solution 3. As in Solution 2, we find that $f(0) = -1$,

$$(f(x)+1)^2 = f(x^2)+1 \tag{8}$$

and

$$f(x) = f(-x) \quad \text{or} \quad f(x)+f(-x) = -2 \tag{9}$$

for any $x \in \mathbb{R}$. We shall show that one of the statements in (9) holds for all $x \in \mathbb{R}$. Suppose $f(a) = f(-a)$ but $f(a)+f(-a) \neq -2$, while $f(b) \neq f(-b)$ but $f(b)+f(-b) = -2$. Clearly, $a, b \neq 0$ and $f(a), f(b) \neq -1$.

Taking $y = a$ and $y = -a$ in (1) respectively and comparing the two equations obtained, we have $f(x+a)^2 = f(x-a)^2$, that is, $f(x+a) = \pm f(x-a)$. This implies $f(x+2a) = \pm f(x)$ for all $x \in \mathbb{R}$. Putting $x = b$ and $x = -2a-b$ respectively, we find $f(2a+b) = \pm f(b)$ and $f(-2a-b) = \pm f(-b) = \pm(-2-f(b))$. Since $f(b) \neq -1$, the term $\pm(-2-f(b))$ is distinct from $\pm f(b)$ in any case. So $f(2a+b) \neq f(-2a-b)$. From (9), we must have $f(2a+b)+f(-2a-b) = -2$. Note that we also have $f(b)+f(-b) = -2$ where $|f(b)|, |f(-b)|$ are equal to $|f(2a+b)|, |f(-2a-b)|$ respectively. The only possible case is $f(2a+b) = f(b)$ and $f(-2a-b) = f(-b)$.

Applying the argument to $-a$ instead of a and using induction, we have $f(2ka+b) = f(b)$ and $f(2ka-b) = f(-b)$ for any integer k. Note that $f(b)+f(-b) = -2$ and $f(b) \neq -1$ imply one of $f(b), f(-b)$ is less than -1. Without loss of generality, assume $f(b) < -1$. We consider $x = \sqrt{2ka+b}$ in (8) for sufficiently large k so that

$$(f(x)+1)^2 = f(2ka+b)+1 = f(b)+1 < 0$$

yields a contradiction. Therefore, one of the statements in (9) must hold for all $x \in \mathbb{R}$.

- **Case 1.** $f(x) = f(-x)$ for any $x \in \mathbb{R}$.

For any $a \in \mathbb{R}$, setting $x = y = \frac{a}{2}$ and $x = -y = \frac{a}{2}$ in (1) respectively and comparing these, we obtain $f(a)^2 = f(0)^2 = 1$, which means $f(a) = \pm 1$ for all $a \in \mathbb{R}$. If $f(a) = 1$ for some a, we may assume $a > 0$ since $f(a) = f(-a)$. Taking $x = y = \sqrt{a}$ in (1), we get

$$f(2\sqrt{a})^2 = 2f(\sqrt{a})^2 + \max\{2, f(2a)\} = 2f(\sqrt{a})^2 + 2.$$

Note that the left-hand side is ± 1 while the right-hand side is an even integer. This is a contradiction. Therefore, $f(x) = -1$ for all $x \in \mathbb{R}$, which is clearly a solution.

- **Case 2.** $f(x) + f(-x) = -2$ for any $x \in \mathbb{R}$.

This case can be handled in the same way as in Solution 2, which yields another solution $f(x) = x - 1$.

A8. Determine the largest real number a such that for all $n \geqslant 1$ and for all real numbers $x_0, x_1, \ldots, x_n$ satisfying $0 = x_0 < x_1 < x_2 < \cdots < x_n$, we have

$$\frac{1}{x_1 - x_0} + \frac{1}{x_2 - x_1} + \cdots + \frac{1}{x_n - x_{n-1}} \geqslant a\left(\frac{2}{x_1} + \frac{3}{x_2} + \cdots + \frac{n+1}{x_n}\right). \tag{1}$$

Answer. The largest a is $\frac{4}{9}$.

Solution 1. We first show that $a = \frac{4}{9}$ is admissible. For each $2 \leqslant k \leqslant n$, by the Cauchy-Schwarz Inequality, we have

$$(x_{k-1} + (x_k - x_{k-1}))\left(\frac{(k-1)^2}{x_{k-1}} + \frac{3^2}{x_k - x_{k-1}}\right) \geqslant (k-1+3)^2,$$

which can be rewritten as

$$\frac{9}{x_k - x_{k-1}} \geqslant \frac{(k+2)^2}{x_k} - \frac{(k-1)^2}{x_{k-1}}. \tag{2}$$

Summing (2) over $k = 2, 3, \ldots, n$ and adding $\frac{9}{x_1}$ to both sides, we have

$$9\sum_{k=1}^{n} \frac{1}{x_k - x_{k-1}} \geqslant 4\sum_{k=1}^{n} \frac{k+1}{x_k} + \frac{n^2}{x_n} > 4\sum_{k=1}^{n} \frac{k+1}{x_k}.$$

This shows (1) holds for $a = \frac{4}{9}$.

Next, we show that $a = \frac{4}{9}$ is the optimal choice. Consider the sequence defined by $x_0 = 0$ and $x_k = x_{k-1} + k(k+1)$ for $k \geqslant 1$, that is, $x_k = \frac{1}{3}k(k+1)(k+2)$. Then the left-hand side of (1) equals

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{n+1},$$

while the right-hand side equals

$$a\sum_{k=1}^{n} \frac{k+1}{x_k} = 3a\sum_{k=1}^{n} \frac{1}{k(k+2)} = \frac{3}{2}a\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+2}\right) = \frac{3}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)a.$$

When n tends to infinity, the left-hand side tends to 1 while the right-hand side tends to $\frac{9}{4}a$. Therefore a has to be at most $\frac{4}{9}$.

Hence the largest value of a is $\frac{4}{9}$.

Solution 2. We shall give an alternative method to establish (1) with $a = \frac{4}{9}$. We define $y_k = x_k - x_{k-1} > 0$ for $1 \leqslant k \leqslant n$. By the Cauchy-Schwarz Inequality, for $1 \leqslant k \leqslant n$, we have

$$(y_1 + y_2 + \cdots + y_k)\left(\sum_{j=1}^{k} \frac{1}{y_j}\binom{j+1}{2}^2\right) \geqslant \left(\binom{2}{2} + \binom{3}{2} + \cdots + \binom{k+1}{2}\right)^2 = \binom{k+2}{3}^2.$$

This can be rewritten as

$$\frac{k+1}{y_1+y_2+\cdots+y_k} \leqslant \frac{36}{k^2(k+1)(k+2)^2}\left(\sum_{j=1}^{k}\frac{1}{y_j}\binom{j+1}{2}^2\right). \tag{3}$$

Summing (3) over $k=1,2,\ldots,n$, we get

$$\frac{2}{y_1}+\frac{3}{y_1+y_2}+\cdots+\frac{n+1}{y_1+y_2+\cdots+y_n} \leqslant \frac{c_1}{y_1}+\frac{c_2}{y_2}+\cdots+\frac{c_n}{y_n} \tag{4}$$

where for $1 \leqslant m \leqslant n$,

$$\begin{aligned} c_m &= 36\binom{m+1}{2}^2 \sum_{k=m}^{n}\frac{1}{k^2(k+1)(k+2)^2} \\ &= \frac{9m^2(m+1)^2}{4}\sum_{k=m}^{n}\left(\frac{1}{k^2(k+1)^2}-\frac{1}{(k+1)^2(k+2)^2}\right) \\ &= \frac{9m^2(m+1)^2}{4}\left(\frac{1}{m^2(m+1)^2}-\frac{1}{(n+1)^2(n+2)^2}\right) < \frac{9}{4}. \end{aligned}$$

From (4), the inequality (1) holds for $a=\frac{4}{9}$. This is also the upper bound as can be verified in the same way as Solution 1.

Combinatorics

C1. The leader of an IMO team chooses positive integers n and k with $n > k$, and announces them to the deputy leader and a contestant. The leader then secretly tells the deputy leader an n-digit binary string, and the deputy leader writes down all n-digit binary strings which differ from the leader's in exactly k positions. (For example, if $n = 3$ and $k = 1$, and if the leader chooses 101, the deputy leader would write down 001, 111 and 100.) The contestant is allowed to look at the strings written by the deputy leader and guess the leader's string. What is the minimum number of guesses (in terms of n and k) needed to guarantee the correct answer?

Answer. The minimum number of guesses is 2 if $n = 2k$ and 1 if $n \neq 2k$.

Solution 1. Let X be the binary string chosen by the leader and let X' be the binary string of length n every digit of which is different from that of X. The strings written by the deputy leader are the same as those in the case when the leader's string is X' and k is changed to $n - k$. In view of this, we may assume $k \geqslant \frac{n}{2}$. Also, for the particular case $k = \frac{n}{2}$, this argument shows that the strings X and X' cannot be distinguished, and hence in that case the contestant has to guess at least twice.

It remains to show that the number of guesses claimed suffices. Consider any string Y which differs from X in m digits where $0 < m < 2k$. Without loss of generality, assume the first m digits of X and Y are distinct. Let Z be the binary string obtained from X by changing its first k digits. Then Z is written by the deputy leader. Note that Z differs from Y by $|m-k|$ digits where $|m-k| < k$ since $0 < m < 2k$. From this observation, the contestant must know that Y is not the desired string.

As we have assumed $k \geqslant \frac{n}{2}$, when $n < 2k$, every string $Y \neq X$ differs from X in fewer than $2k$ digits. When $n = 2k$, every string except X and X' differs from X in fewer than $2k$ digits. Hence, the answer is as claimed.

Solution 2. Firstly, assume $n \neq 2k$. Without loss of generality suppose the first digit of the leader's string is 1. Then among the $\binom{n}{k}$ strings written by the deputy leader, $\binom{n-1}{k}$ will begin with 1 and $\binom{n-1}{k-1}$ will begin with 0. Since $n \neq 2k$, we have $k + (k-1) \neq n-1$ and so $\binom{n-1}{k} \neq \binom{n-1}{k-1}$. Thus, by counting the number of strings written by the deputy leader that start with 0 and 1, the contestant can tell the first digit of the leader's string. The same can be done on the other digits, so 1 guess suffices when $n \neq 2k$.

Secondly, for the case $n = 2$ and $k = 1$, the answer is clearly 2. For the remaining cases where $n = 2k > 2$, the deputy leader would write down the same strings if the leader's string X is replaced by X' obtained by changing each digit of X. This shows at least 2 guesses are needed. We shall show that 2 guesses suffice in this case. Suppose the first two digits of the leader's string are the same. Then among the strings written by the deputy leader, the prefices 01 and 10 will occur $\binom{2k-2}{k-1}$ times each, while the prefices 00 and 11 will occur $\binom{2k-2}{k}$ times each. The two numbers are interchanged if the first two digits of the leader's string are different. Since $\binom{2k-2}{k-1} \neq \binom{2k-2}{k}$, the contestant can tell whether the first two digits of the leader's string are the same or not. He can work out the relation of the first digit and the other digits in the same way and reduce the leader's string to only 2 possibilities. The proof is complete.

C2. Find all positive integers n for which all positive divisors of n can be put into the cells of a rectangular table under the following constraints:

- each cell contains a distinct divisor;
- the sums of all rows are equal; and
- the sums of all columns are equal.

Answer. 1.

Solution 1. Suppose all positive divisors of n can be arranged into a rectangular table of size $k \times l$ where the number of rows k does not exceed the number of columns l. Let the sum of numbers in each column be s. Since n belongs to one of the columns, we have $s \geqslant n$, where equality holds only when $n = 1$.

For $j = 1, 2, \ldots, l$, let d_j be the largest number in the j-th column. Without loss of generality, assume $d_1 > d_2 > \cdots > d_l$. Since these are divisors of n, we have

$$d_l \leqslant \frac{n}{l}. \tag{1}$$

As d_l is the maximum entry of the l-th column, we must have

$$d_l \geqslant \frac{s}{k} \geqslant \frac{n}{k}. \tag{2}$$

The relations (1) and (2) combine to give $\frac{n}{l} \geqslant \frac{n}{k}$, that is, $k \geqslant l$. Together with $k \leqslant l$, we conclude that $k = l$. Then all inequalities in (1) and (2) are equalities. In particular, $s = n$ and so $n = 1$, in which case the conditions are clearly satisfied.

Solution 2. Clearly $n = 1$ works. Then we assume $n > 1$ and let its prime factorization be $n = p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t}$. Suppose the table has k rows and l columns with $1 < k \leqslant l$. Note that kl is the number of positive divisors of n and the sum of all entries is the sum of positive divisors of n, which we denote by $\sigma(n)$. Consider the column containing n. Since the column sum is $\frac{\sigma(n)}{l}$, we must have $\frac{\sigma(n)}{l} > n$. Therefore, we have

$$\begin{aligned}(r_1+1)(r_2+1)\cdots(r_t+1) = kl \leqslant l^2 &< \left(\frac{\sigma(n)}{n}\right)^2 \\ &= \left(1 + \frac{1}{p_1} + \cdots + \frac{1}{p_1^{r_1}}\right)^2 \cdots \left(1 + \frac{1}{p_t} + \cdots + \frac{1}{p_t^{r_t}}\right)^2.\end{aligned}$$

This can be rewritten as

$$f(p_1, r_1) f(p_2, r_2) \cdots f(p_t, r_t) < 1 \tag{3}$$

where

$$f(p, r) = \frac{r+1}{\left(1 + \frac{1}{p} + \cdots + \frac{1}{p^r}\right)^2} = \frac{(r+1)\left(1 - \frac{1}{p}\right)^2}{\left(1 - \frac{1}{p^{r+1}}\right)^2}.$$

Direct computation yields

$$f(2,1)=\frac{8}{9},\quad f(2,2)=\frac{48}{49},\quad f(3,1)=\frac{9}{8}.$$

Also, we find that

$$\begin{aligned}
f(2,r) &\geqslant \left(1-\frac{1}{2^{r+1}}\right)^{-2}>1 \quad \text{for } r\geqslant 3,\\
f(3,r) &\geqslant \frac{4}{3}\left(1-\frac{1}{3^{r+1}}\right)^{-2}>\frac{4}{3}>\frac{9}{8} \quad \text{for } r\geqslant 2, \text{ and}\\
f(p,r) &\geqslant \frac{32}{25}\left(1-\frac{1}{p^{r+1}}\right)^{-2}>\frac{32}{25}>\frac{9}{8} \quad \text{for } p\geqslant 5.
\end{aligned}$$

From these values and bounds, it is clear that (3) holds only when $n=2$ or 4. In both cases, it is easy to see that the conditions are not satisfied. Hence, the only possible n is 1.

C3. Let n be a positive integer relatively prime to 6. We paint the vertices of a regular n-gon with three colours so that there is an odd number of vertices of each colour. Show that there exists an isosceles triangle whose three vertices are of different colours.

Solution. For $k = 1, 2, 3$, let a_k be the number of isosceles triangles whose vertices contain exactly k colours. Suppose on the contrary that $a_3 = 0$. Let b, c, d be the number of vertices of the three different colours respectively. We now count the number of pairs $(\triangle, E)$ where $\triangle$ is an isosceles triangle and E is a side of $\triangle$ whose endpoints are of different colours.

On the one hand, since we have assumed $a_3 = 0$, each triangle in the pair must contain exactly two colours, and hence each triangle contributes twice. Thus the number of pairs is $2a_2$.

On the other hand, if we pick any two vertices A, B of distinct colours, then there are three isosceles triangles having these as vertices, two when AB is not the base and one when AB is the base since n is odd. Note that the three triangles are all distinct as $(n, 3) = 1$. In this way, we count the number of pairs to be $3(bc + cd + db)$. However, note that $2a_2$ is even while $3(bc+cd+db)$ is odd, as each of b, c, d is. This yields a contradiction and hence $a_3 \geqslant 1$.

Comment. A slightly stronger version of this problem is to replace the condition $(n, 6) = 1$ by n being odd (where equilateral triangles are regarded as isosceles triangles). In that case, the only difference in the proof is that by fixing any two vertices A, B, one can find exactly one or three isosceles triangles having these as vertices. But since only parity is concerned in the solution, the proof goes the same way.

The condition that there is an odd number of vertices of each colour is necessary, as can be seen from the following example. Consider $n = 25$ and we label the vertices $A_0, A_1, \ldots, A_{24}$. Suppose colour 1 is used for A_0, colour 2 is used for $A_5, A_{10}, A_{15}, A_{20}$, while colour 3 is used for the remaining vertices. Then any isosceles triangle having colours 1 and 2 must contain A_0 and one of $A_5, A_{10}, A_{15}, A_{20}$. Clearly, the third vertex must have index which is a multiple of 5 so it is not of colour 3.

C4. Find all positive integers n for which we can fill in the entries of an $n \times n$ table with the following properties:

- each entry can be one of I, M and O;
- in each row and each column, the letters I, M and O occur the same number of times; and
- in any diagonal whose number of entries is a multiple of three, the letters I, M and O occur the same number of times.

Answer. n can be any multiple of 9.

Solution. We first show that such a table exists when n is a multiple of 9. Consider the following 9×9 table.

$$\begin{pmatrix} I & I & I & M & M & M & O & O & O \\ M & M & M & O & O & O & I & I & I \\ O & O & O & I & I & I & M & M & M \\ I & I & I & M & M & M & O & O & O \\ M & M & M & O & O & O & I & I & I \\ O & O & O & I & I & I & M & M & M \\ I & I & I & M & M & M & O & O & O \\ M & M & M & O & O & O & I & I & I \\ O & O & O & I & I & I & M & M & M \end{pmatrix} \tag{1}$$

It is a direct checking that the table (1) satisfies the requirements. For $n = 9k$ where k is a positive integer, we form an $n \times n$ table using $k \times k$ copies of (1). For each row and each column of the table of size n, since there are three I's, three M's and three O's for any nine consecutive entries, the numbers of I, M and O are equal. In addition, every diagonal of the large table whose number of entries is divisible by 3 intersects each copy of (1) at a diagonal with number of entries divisible by 3 (possibly zero). Therefore, every such diagonal also contains the same number of I, M and O.

Next, consider any $n \times n$ table for which the requirements can be met. As the number of entries of each row should be a multiple of 3, we let $n = 3k$ where k is a positive integer. We divide the whole table into $k \times k$ copies of 3×3 blocks. We call the entry at the centre of such a 3×3 square a *vital entry*. We also call any row, column or diagonal that contains at least one vital entry a *vital line*. We compute the number of pairs (l, c) where l is a vital line and c is an entry belonging to l that contains the letter M. We let this number be N.

On the one hand, since each vital line contains the same number of I, M and O, it is obvious that each vital row and each vital column contain k occurrences of M. For vital diagonals in either direction, we count there are exactly

$$1 + 2 + \cdots + (k-1) + k + (k-1) + \cdots + 2 + 1 = k^2$$

occurrences of M. Therefore, we have $N = 4k^2$.

On the other hand, there are $3k^2$ occurrences of M in the whole table. Note that each entry belongs to exactly 1 or 4 vital lines. Therefore, N must be congruent to $3k^2$ mod 3.

From the double counting, we get $4k^2 \equiv 3k^2 \pmod 3$, which forces k to be a multiple of 3. Therefore, n has to be a multiple of 9 and the proof is complete.

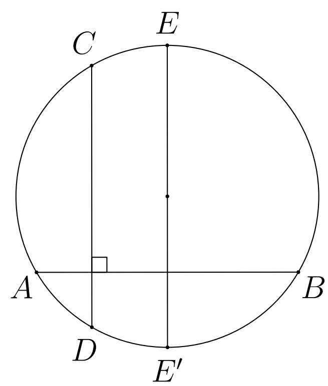
C5. Let $n \geqslant 3$ be a positive integer. Find the maximum number of diagonals of a regular n-gon one can select, so that any two of them do not intersect in the interior or they are perpendicular to each other.

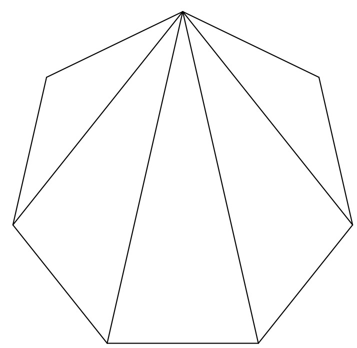
Answer. $n-2$ if n is even and $n-3$ if n is odd.

Solution 1. We consider two cases according to the parity of n.

- **Case 1.** n is odd.

We first claim that no pair of diagonals is perpendicular. Suppose A, B, C, D are vertices where AB and CD are perpendicular, and let E be the vertex lying on the perpendicular bisector of AB. Let E' be the opposite point of E on the circumcircle of the regular polygon. Since $EC = E'D$ and C, D, E are vertices of the regular polygon, E' should also belong to the polygon. This contradicts the fact that a regular polygon with an odd number of vertices does not contain opposite points on the circumcircle.



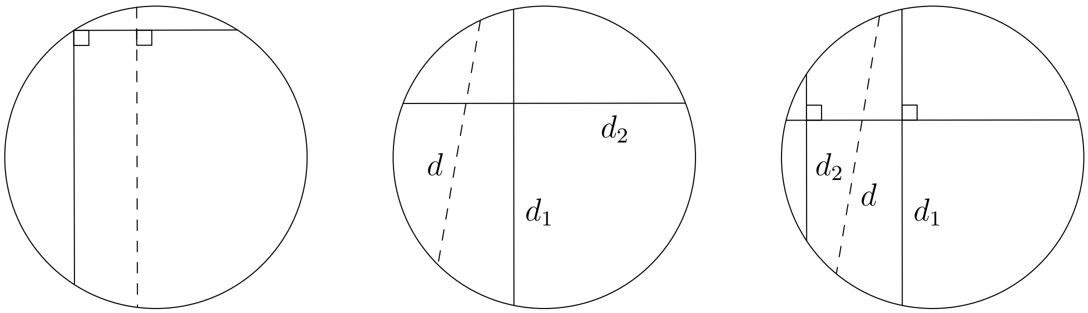


Therefore in the odd case we can only select diagonals which do not intersect. In the maximal case these diagonals should divide the regular n-gon into $n-2$ triangles, so we can select at most $n-3$ diagonals. This can be done, for example, by selecting all diagonals emanated from a particular vertex.

- **Case 2.** n is even.

If there is no intersection, then the proof in the odd case works. Suppose there are two perpendicular diagonals selected. We consider the set S of all selected diagonals parallel to one of them which intersect with some selected diagonals. Suppose S contains k diagonals and the number of distinct endpoints of the k diagonals is l.

Firstly, consider the longest diagonal in one of the two directions in S. No other diagonal in S can start from either endpoint of that diagonal, since otherwise it has to meet another longer diagonal in S. The same holds true for the other direction. Ignoring these two longest diagonals and their four endpoints, the remaining $k-2$ diagonals share $l-4$ endpoints where each endpoint can belong to at most two diagonals. This gives $2(l-4) \geqslant 2(k-2)$, so that $k \leqslant l-2$.

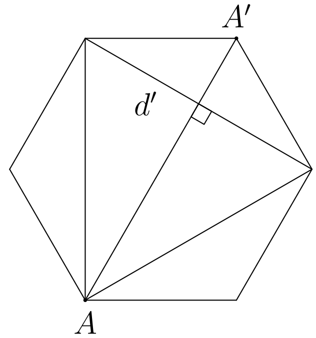


Consider a group of consecutive vertices of the regular n-gon so that each of the two outermost vertices is an endpoint of a diagonal in S, while the interior points are not. There are l such groups. We label these groups $P_1, P_2, \ldots, P_l$ in this order. We claim that each selected diagonal outside S must connect vertices of the same group P_i. Consider any diagonal d joining vertices from distinct groups P_i and P_j. Let d_1 and d_2 be two diagonals in S each having one of the outermost points of P_i as endpoint. Then d must meet either d_1, d_2 or a diagonal in S which is perpendicular to both d_1 and d_2. In any case d should belong to S by definition, which is a contradiction.

Within the same group P_i, there are no perpendicular diagonals since the vertices belong to the same side of a diameter of the circumcircle. Hence there can be at most $|P_i|-2$ selected diagonals within P_i, including the one joining the two outermost points of P_i when $|P_i| > 2$. Therefore, the maximum number of diagonals selected is

$$\sum_{i=1}^{l}(|P_i|-2)+k=\sum_{i=1}^{l}|P_i|-2l+k=(n+l)-2l+k=n-l+k \leqslant n-2.$$

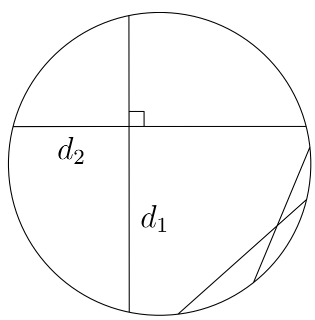
This upper bound can be attained as follows. We take any vertex A and let A' be the vertex for which AA' is a diameter of the circumcircle. If we select all diagonals emanated from A together with the diagonal d' joining the two neighbouring vertices of A', then the only pair of diagonals that meet each other is AA' and d', which are perpendicular to each other. In total we can take $n-2$ diagonals.

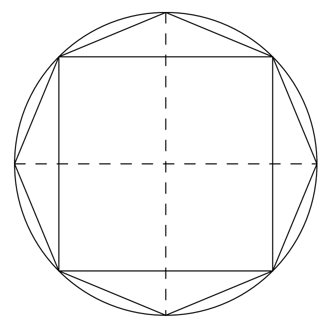


Solution 2. The constructions and the odd case are the same as Solution 1. Instead of dealing separately with the case where n is even, we shall prove by induction more generally that we can select at most $n-2$ diagonals for any cyclic n-gon with circumcircle Γ.

The base case $n = 3$ is trivial since there is no diagonal at all. Suppose the upper bound holds for any cyclic polygon with fewer than n sides. For a cyclic n-gon, if there is a selected diagonal which does not intersect any other selected diagonal, then this diagonal divides the n-gon into an m-gon and an l-gon (with $m+l = n+2$) so that each selected diagonal belongs to one of them. Without loss of generality, we may assume the m-gon lies on the same side of a diameter of Γ. Then no two selected diagonals of the m-gon can intersect, and hence we can select at most $m-3$ diagonals. Also, we can apply the inductive hypothesis to the l-gon. This shows the maximum number of selected diagonals is $(m-3)+(l-2)+1 = n-2$.

It remains to consider the case when all selected diagonals meet at least one other selected diagonal. Consider a pair of selected perpendicular diagonals d_1, d_2. They divide the circumference of Γ into four arcs, each of which lies on the same side of a diameter of Γ. If there are two selected diagonals intersecting each other and neither is parallel to d_1 or d_2, then their endpoints must belong to the same arc determined by d_1, d_2, and hence they cannot be perpendicular. This violates the condition, and hence all selected diagonals must have the same direction as one of d_1, d_2.





Take the longest selected diagonal in one of the two directions. We argue as in Solution 1 that its endpoints do not belong to any other selected diagonal. The same holds true for the longest diagonal in the other direction. Apart from these four endpoints, each of the remaining $n-4$ vertices can belong to at most two selected diagonals. Thus we can select at most $\frac{1}{2}(2(n-4)+4) = n-2$ diagonals. Then the proof follows by induction.

C6. There are $n \geqslant 3$ islands in a city. Initially, the ferry company offers some routes between some pairs of islands so that it is impossible to divide the islands into two groups such that no two islands in different groups are connected by a ferry route.

After each year, the ferry company will close a ferry route between some two islands X and Y. At the same time, in order to maintain its service, the company will open new routes according to the following rule: for any island which is connected by a ferry route to exactly one of X and Y, a new route between this island and the other of X and Y is added.

Suppose at any moment, if we partition all islands into two nonempty groups in any way, then it is known that the ferry company will close a certain route connecting two islands from the two groups after some years. Prove that after some years there will be an island which is connected to all other islands by ferry routes.

Solution. Initially, we pick any pair of islands A and B which are connected by a ferry route and put A in set $\mathcal{A}$ and B in set $\mathcal{B}$. From the condition, without loss of generality there must be another island which is connected to A. We put such an island C in set $\mathcal{B}$. We say that two sets of islands form a *network* if each island in one set is connected to each island in the other set.

Next, we shall included all islands to $\mathcal{A} \cup \mathcal{B}$ one by one. Suppose we have two sets $\mathcal{A}$ and $\mathcal{B}$ which form a network where $3 \leqslant |\mathcal{A} \cup \mathcal{B}| < n$. This relation no longer holds only when a ferry route between islands $A \in \mathcal{A}$ and $B \in \mathcal{B}$ is closed. In that case, we define $\mathcal{A}' = \{A, B\}$, and $\mathcal{B}' = (\mathcal{A} \cup \mathcal{B}) - \{A, B\}$. Note that $\mathcal{B}'$ is nonempty. Consider any island $C \in \mathcal{A} - \{A\}$. From the relation of $\mathcal{A}$ and $\mathcal{B}$, we know that C is connected to B. If C was not connected to A before the route between A and B closes, then there will be a route added between C and A afterwards. Hence, C must now be connected to both A and B. The same holds true for any island in $\mathcal{B} - \{B\}$. Therefore, $\mathcal{A}'$ and $\mathcal{B}'$ form a network, and $\mathcal{A}' \cup \mathcal{B}' = \mathcal{A} \cup \mathcal{B}$. Hence these islands can always be partitioned into sets $\mathcal{A}$ and $\mathcal{B}$ which form a network.

As $|\mathcal{A} \cup \mathcal{B}| < n$, there are some islands which are not included in $\mathcal{A} \cup \mathcal{B}$. From the condition, after some years there must be a ferry route between an island A in $\mathcal{A} \cup \mathcal{B}$ and an island D outside $\mathcal{A} \cup \mathcal{B}$ which closes. Without loss of generality assume $A \in \mathcal{A}$. Then each island in $\mathcal{B}$ must then be connected to D, no matter it was or not before. Hence, we can put D in set $\mathcal{A}$ so that the new sets $\mathcal{A}$ and $\mathcal{B}$ still form a network and the size of $\mathcal{A} \cup \mathcal{B}$ is increased by 1. The same process can be done to increase the size of $\mathcal{A} \cup \mathcal{B}$. Eventually, all islands are included in this way so we may now assume $|\mathcal{A} \cup \mathcal{B}| = n$.

Suppose a ferry route between $A \in \mathcal{A}$ and $B \in \mathcal{B}$ is closed after some years. We put A and B in set $\mathcal{A}'$ and all remaining islands in set $\mathcal{B}'$. Then $\mathcal{A}'$ and $\mathcal{B}'$ form a network. This relation no longer holds only when a route between A, without loss of generality, and $C \in \mathcal{B}'$ is closed. Since this must eventually occur, at that time island B will be connected to all other islands and the result follows.

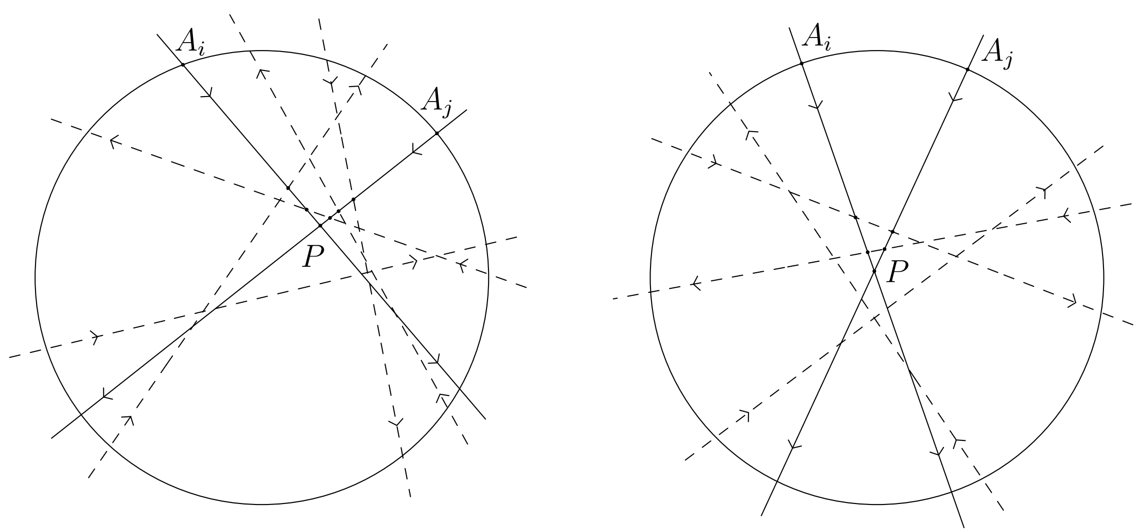
C7. Let $n \geqslant 2$ be an integer. In the plane, there are n segments given in such a way that any two segments have an intersection point in the interior, and no three segments intersect at a single point. Jeff places a snail at one of the endpoints of each of the segments and claps his hands $n-1$ times. Each time when he claps his hands, all the snails move along their own segments and stay at the next intersection points until the next clap. Since there are $n-1$ intersection points on each segment, all snails will reach the furthest intersection points from their starting points after $n-1$ claps.

(a) Prove that if n is odd then Jeff can always place the snails so that no two of them ever occupy the same intersection point.

(b) Prove that if n is even then there must be a moment when some two snails occupy the same intersection point no matter how Jeff places the snails.

Solution. We consider a big disk which contains all the segments. We extend each segment to a line l_i so that each of them cuts the disk at two distinct points A_i, B_i.

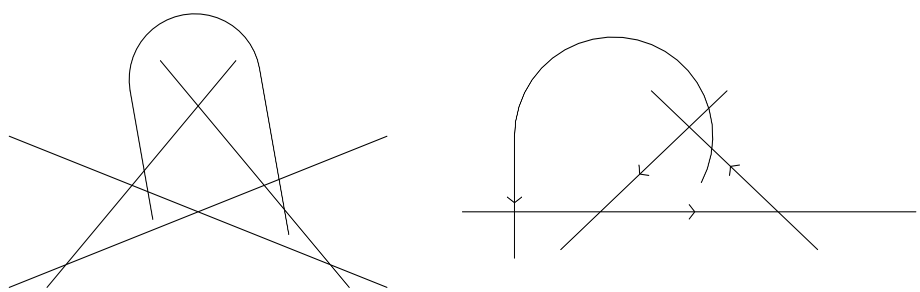
(a) For odd n, we travel along the circumference of the disk and mark each of the points A_i or B_i 'in' and 'out' alternately. Since every pair of lines intersect in the disk, there are exactly $n-1$ points between A_i and B_i for any fixed $1 \leqslant i \leqslant n$. As n is odd, this means one of A_i and B_i is marked 'in' and the other is marked 'out'. Then Jeff can put a snail on the endpoint of each segment which is closer to the 'in' side of the corresponding line. We claim that the snails on l_i and l_j do not meet for any pairs i, j, hence proving part (a).



Without loss of generality, we may assume the snails start at A_i and A_j respectively. Let l_i intersect l_j at P. Note that there is an odd number of points between arc A_iA_j. Each of these points belongs to a line l_k. Such a line l_k must intersect exactly one of the segments A_iP and A_jP, making an odd number of intersections. For the other lines, they may intersect both segments A_iP and A_jP, or meet none of them. Therefore, the total number of intersection points on segments A_iP and A_jP (not counting P) is odd. However, if the snails arrive at P at the same time, then there should be the same number of intersections on A_iP and A_jP, which gives an even number of intersections. This is a contradiction so the snails do not meet each other.

(b) For even n, we consider any way that Jeff places the snails and mark each of the points A_i or B_i 'in' and 'out' according to the directions travelled by the snails. In this case there must be two neighbouring points A_i and A_j both of which are marked 'in'. Let P be the intersection of the segments A_iB_i and A_jB_j. Then any other segment meeting one of the segments A_iP and A_jP must also meet the other one, and so the number of intersections on A_iP and A_jP are the same. This shows the snails starting from A_i and A_j will meet at P.

Comment. The conclusions do not hold for pseudosegments, as can be seen from the following examples.

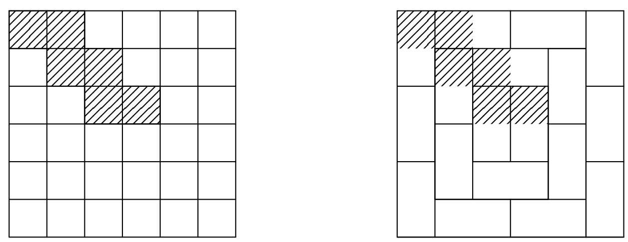


C8. Let n be a positive integer. Determine the smallest positive integer k with the following property: it is possible to mark k cells on a $2n \times 2n$ board so that there exists a unique partition of the board into 1×2 and 2×1 dominoes, none of which contains two marked cells.

Answer. $2n$.

Solution. We first construct an example of marking $2n$ cells satisfying the requirement. Label the rows and columns $1, 2, \ldots, 2n$ and label the cell in the i-th row and the j-th column (i, j).

For $i = 1, 2, \ldots, n$, we mark the cells (i, i) and $(i, i + 1)$. We claim that the required partition exists and is unique. The two diagonals of the board divide the board into four regions. Note that the domino covering cell $(1, 1)$ must be vertical. This in turn shows that each domino covering $(2, 2), (3, 3), \ldots, (n, n)$ is vertical. By induction, the dominoes in the left region are all vertical. By rotational symmetry, the dominoes in the bottom region are horizontal, and so on. This shows that the partition exists and is unique.



It remains to show that this value of k is the smallest possible. Assume that only $k < 2n$ cells are marked, and there exists a partition P satisfying the requirement. It suffices to show there exists another desirable partition distinct from P. Let d be the main diagonal of the board.

Construct the following graph with edges of two colours. Its vertices are the cells of the board. Connect two vertices with a red edge if they belong to the same domino of P. Connect two vertices with a blue edge if their reflections in d are connected by a red edge. It is possible that two vertices are connected by edges of both colours. Clearly, each vertex has both red and blue degrees equal to 1. Thus the graph splits into cycles where the colours of edges in each cycle alternate (a cycle may have length 2).

Consider any cell c lying on the diagonal d. Its two edges are symmetrical with respect to d. Thus they connect c to different cells. This shows c belongs to a cycle $C(c)$ of length at least 4. Consider a part of this cycle $c_0, c_1, \ldots, c_m$ where $c_0 = c$ and m is the least positive integer such that c_m lies on d. Clearly, c_m is distinct from c. From the construction, the path symmetrical to this with respect to d also lies in the graph, and so these paths together form $C(c)$. Hence, $C(c)$ contains exactly two cells from d. Then all $2n$ cells in d belong to n cycles $C_1, C_2, \ldots, C_n$, each has length at least 4.

By the Pigeonhole Principle, there exists a cycle C_i containing at most one of the k marked cells. We modify P as follows. We remove all dominoes containing the vertices of C_i, which correspond to the red edges of C_i. Then we put the dominoes corresponding to the blue edges of C_i. Since C_i has at least 4 vertices, the resultant partition P' is different from P. Clearly, no domino in P' contains two marked cells as C_i contains at most one marked cell. This shows the partition is not unique and hence k cannot be less than $2n$.

Geometry

G1. In a convex pentagon $ABCDE$, let F be a point on AC such that $\angle FBC = 90^\circ$. Suppose triangles ABF, ACD and ADE are similar isosceles triangles with

$$\angle FAB = \angle FBA = \angle DAC = \angle DCA = \angle EAD = \angle EDA. \tag{1}$$

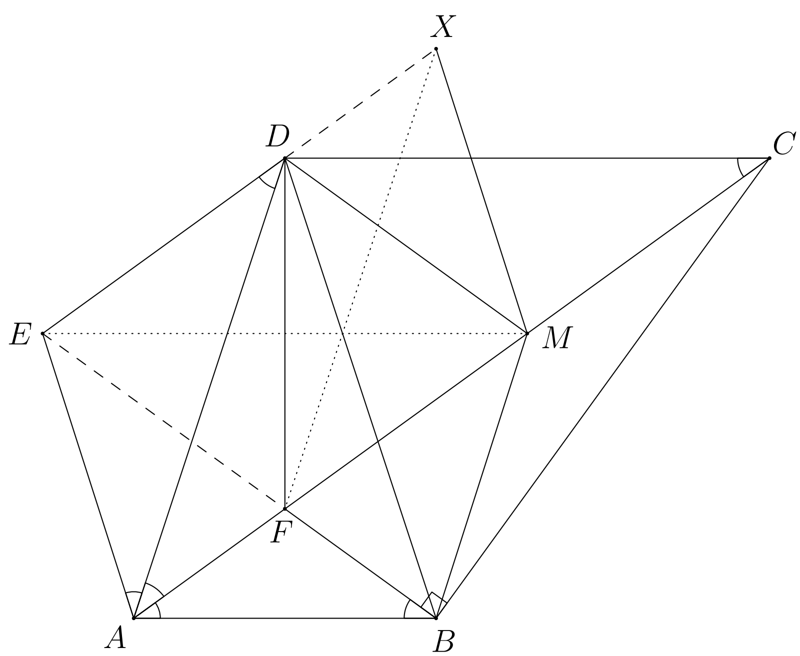
Let M be the midpoint of CF. Point X is chosen such that $AMXE$ is a parallelogram. Show that BD, EM and FX are concurrent.

Solution 1. Denote the common angle in (1) by θ. As $\triangle ABF \sim \triangle ACD$, we have $\frac{AB}{AC} = \frac{AF}{AD}$ so that $\triangle ABC \sim \triangle AFD$. From $EA = ED$, we get

$$\angle AFD = \angle ABC = 90^\circ + \theta = 180^\circ - \frac{1}{2}\angle AED.$$

Hence, F lies on the circle with centre E and radius EA. In particular, $EF = EA = ED$. As $\angle EFA = \angle EAF = 2\theta = \angle BFC$, points B, F, E are collinear.

As $\angle EDA = \angle MAD$, we have $ED // AM$ and hence E, D, X are collinear. As M is the midpoint of CF and $\angle CBF = 90^\circ$, we get $MF = MB$. In the isosceles triangles EFA and MFB, we have $\angle EFA = \angle MFB$ and $AF = BF$. Therefore, they are congruent to each other. Then we have $BM = AE = XM$ and $BE = BF + FE = AF + FM = AM = EX$. This shows $\triangle EMB \cong \triangle EMX$. As F and D lie on EB and EX respectively and $EF = ED$, we know that lines BD and XF are symmetric with respect to EM. It follows that the three lines are concurrent.



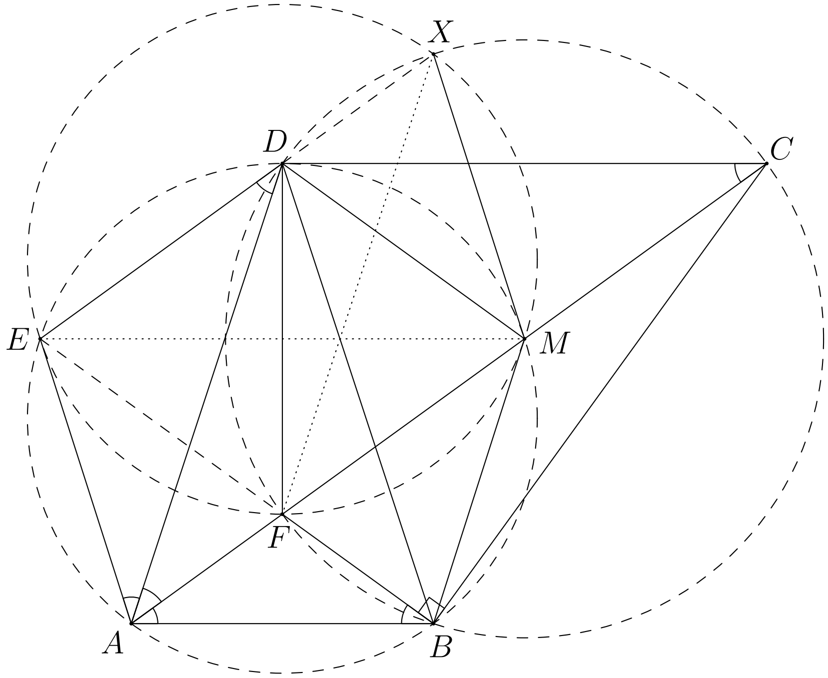
Solution 2. From $\angle CAD = \angle EDA$, we have $AC // ED$. Together with $AC // EX$, we know that E, D, X are collinear. Denote the common angle in (1) by θ. From $\triangle ABF \sim \triangle ACD$, we get $\frac{AB}{AC} = \frac{AF}{AD}$ so that $\triangle ABC \sim \triangle AFD$. This yields $\angle AFD = \angle ABC = 90^\circ + \theta$ and hence $\angle FDC = 90^\circ$, implying that $BCDF$ is cyclic. Let Γ_1 be its circumcircle.

Next, from $\triangle ABF \sim \triangle ADE$, we have $\frac{AB}{AD} = \frac{AF}{AE}$ so that $\triangle ABD \sim \triangle AFE$. Therefore,

$$\angle AFE = \angle ABD = \theta + \angle FBD = \theta + \angle FCD = 2\theta = 180^\circ - \angle BFA.$$

This implies B, F, E are collinear. Note that F is the incentre of triangle DAB. Point E lies on the internal angle bisector of $\angle DBA$ and lies on the perpendicular bisector of AD. It follows that E lies on the circumcircle Γ_2 of triangle ABD, and $EA = EF = ED$.

Also, since CF is a diameter of Γ_1 and M is the midpoint of CF, M is the centre of Γ_1 and hence $\angle AMD = 2\theta = \angle ABD$. This shows M lies on Γ_2. Next, $\angle MDX = \angle MAE = \angle DXM$ since $AMXE$ is a parallelogram. Hence $MD = MX$ and X lies on Γ_1.



We now have two ways to complete the solution.

- **Method 1.** From $EF = EA = XM$ and $EX // FM$, $EFMX$ is an isosceles trapezoid and is cyclic. Denote its circumcircle by Γ_3. Since BD, EM, FX are the three radical axes of $\Gamma_1, \Gamma_2, \Gamma_3$, they must be concurrent.
- **Method 2.** As $\angle DMF = 2\theta = \angle BFM$, we have $DM // EB$. Also,

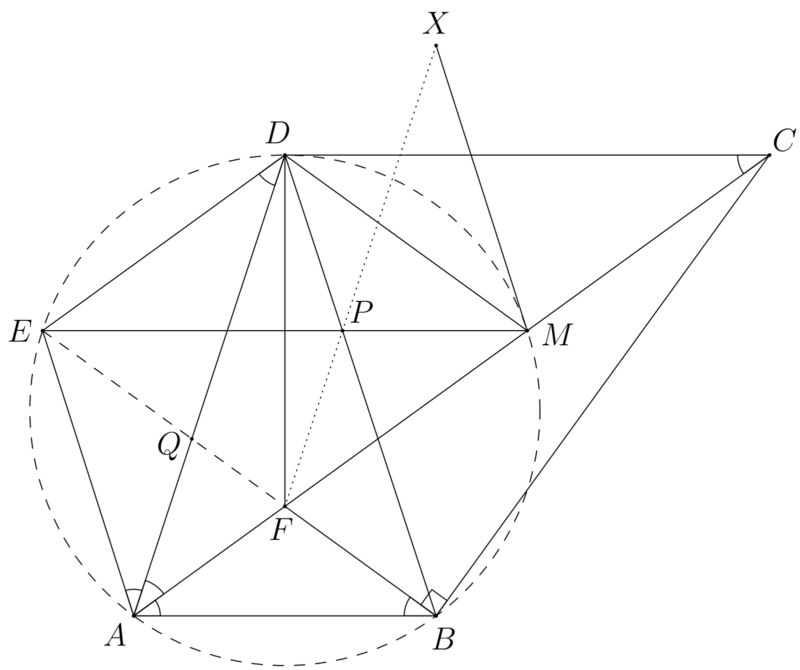
$$\angle BFD + \angle XBF = \angle BFC + \angle CFD + 90^\circ - \angle CBX = 2\theta + (90^\circ - \theta) + 90^\circ - \theta = 180^\circ$$

implies $DF // XB$. These show the corresponding sides of triangles DMF and BEX are parallel. By Desargues' Theorem, the two triangles are perspective and hence DB, ME, FX meet at a point.

Comment. In Solution 2, both the Radical Axis Theorem and Desargues' Theorem could imply DB, ME, FX are parallel. However, this is impossible as can be seen from the configuration. For example, it is obvious that DB and ME meet each other.

Solution 3. Let the common angle in (1) be θ. From $\triangle ABF \sim \triangle ACD$, we have $\frac{AB}{AC} = \frac{AF}{AD}$ so that $\triangle ABC \sim \triangle AFD$. Then $\angle ADF = \angle ACB = 90^\circ - 2\theta = 90^\circ - \angle BAD$ and hence $DF \perp AB$. As $FA = FB$, this implies $\triangle DAB$ is isosceles with $DA = DB$. Then F is the incentre of $\triangle DAB$.

Next, from $\angle AED = 180^\circ - 2\theta = 180^\circ - \angle DBA$, points A, B, D, E are concyclic. Since we also have $EA = ED$, this shows E, F, B are collinear and $EA = EF = ED$.



Note that C lies on the internal angle bisector of $\angle BAD$ and lies on the external angle bisector of $\angle DBA$. It follows that it is the A-excentre of triangle DAB. As M is the midpoint of CF, M lies on the circumcircle of triangle DAB and it is the centre of the circle passing through D, F, B, C. By symmetry, $DEFM$ is a rhombus. Then the midpoints of AX, EM and DF coincide, and it follows that $DAFX$ is a parallelogram.

Let P be the intersection of BD and EM, and Q be the intersection of AD and BE. From $\angle BAC = \angle DCA$, we know that DC, AB, EM are parallel. Thus we have $\frac{DP}{PB} = \frac{CM}{MA}$. This is further equal to $\frac{AE}{BE}$ since $CM = DM = DE = AE$ and $MA = BE$. From $\triangle AEQ \sim \triangle BEA$, we find that

$$\frac{DP}{PB} = \frac{AE}{BE} = \frac{AQ}{BA} = \frac{QF}{FB}$$

by the Angle Bisector Theorem. This implies $QD // FP$ and hence F, P, X are collinear, as desired.

G2. Let ABC be a triangle with circumcircle Γ and incentre I. Let M be the midpoint of side BC. Denote by D the foot of perpendicular from I to side BC. The line through I perpendicular to AI meets sides AB and AC at F and E respectively. Suppose the circumcircle of triangle AEF intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.

Solution 1. Let AM meet Γ again at Y and XY meet BC at D'. It suffices to show $D' = D$. We shall apply the following fact.

- **Claim.** For any cyclic quadrilateral $PQRS$ whose diagonals meet at T, we have

$$\frac{QT}{TS} = \frac{PQ \cdot QR}{PS \cdot SR}.$$

Proof. We use $[W_1W_2W_3]$ to denote the area of $W_1W_2W_3$. Then

$$\frac{QT}{TS} = \frac{[PQR]}{[PSR]} = \frac{\frac{1}{2}PQ \cdot QR \sin \angle PQR}{\frac{1}{2}PS \cdot SR \sin \angle PSR} = \frac{PQ \cdot QR}{PS \cdot SR}.$$

□

Applying the Claim to $ABYC$ and $XBYC$ respectively, we have $1 = \frac{BM}{MC} = \frac{AB \cdot BY}{AC \cdot CY}$ and $\frac{BD'}{D'C} = \frac{XB \cdot BY}{XC \cdot CY}$. These combine to give

$$\frac{BD'}{CD'} = \frac{XB}{XC} \cdot \frac{BY}{CY} = \frac{XB}{XC} \cdot \frac{AC}{AB}. \tag{1}$$

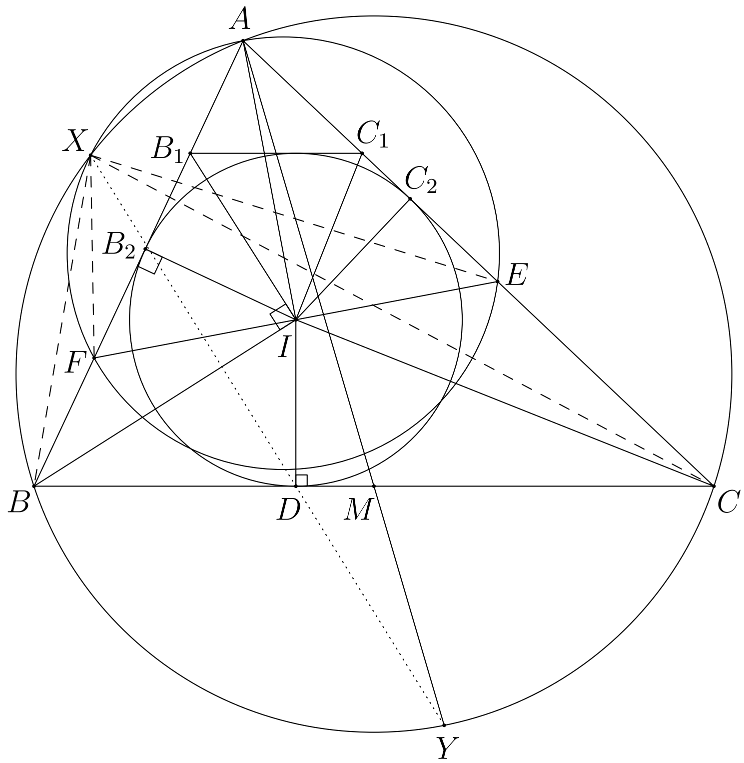
Next, we use directed angles to find that $\measuredangle XBF = \measuredangle XBA = \measuredangle XCA = \measuredangle XCE$ and $\measuredangle XFB = \measuredangle XFA = \measuredangle XEA = \measuredangle XEC$. This shows triangles XBF and XCE are directly similar. In particular, we have

$$\frac{XB}{XC} = \frac{BF}{CE}. \tag{2}$$

In the following, we give two ways to continue the proof.

- **Method 1.** Here is a geometrical method. As $\angle FIB = \angle AIB - 90^\circ = \frac{1}{2}\angle ACB = \angle ICB$ and $\angle FBI = \angle IBC$, the triangles FBI and IBC are similar. Analogously, triangles EIC and IBC are also similar. Hence, we get

$$\frac{FB}{IB} = \frac{BI}{BC} \quad \text{and} \quad \frac{EC}{IC} = \frac{IC}{BC}. \tag{3}$$



Next, construct a line parallel to BC and tangent to the incircle. Suppose it meets sides AB and AC at B_1 and C_1 respectively. Let the incircle touch AB and AC at B_2 and C_2 respectively. By homothety, the line B_1I is parallel to the external angle bisector of $\angle ABC$, and hence $\angle B_1IB = 90^\circ$. Since $\angle BB_2I = 90^\circ$, we get $BB_2 \cdot BB_1 = BI^2$, and similarly $CC_2 \cdot CC_1 = CI^2$. Hence,

$$\frac{BI^2}{CI^2} = \frac{BB_2 \cdot BB_1}{CC_2 \cdot CC_1} = \frac{BB_1}{CC_1} \cdot \frac{BD}{CD} = \frac{AB}{AC} \cdot \frac{BD}{CD}. \tag{4}$$

Combining (1), (2), (3) and (4), we conclude

$$\frac{BD'}{CD'} = \frac{XB}{XC} \cdot \frac{AC}{AB} = \frac{BF}{CE} \cdot \frac{AC}{AB} = \frac{BI^2}{CI^2} \cdot \frac{AC}{AB} = \frac{BD}{CD}$$

so that $D' = D$. The result then follows.

• **Method 2.** We continue the proof of Solution 1 using trigonometry. Let $\beta = \frac{1}{2}\angle ABC$ and $\gamma = \frac{1}{2}\angle ACB$. Observe that $\angle FIB = \angle AIB - 90^\circ = \gamma$. Hence, $\frac{BF}{FI} = \frac{\sin\angle FIB}{\sin\angle IBF} = \frac{\sin\gamma}{\sin\beta}$. Similarly, $\frac{CE}{EI} = \frac{\sin\beta}{\sin\gamma}$. As $FI = EI$, we get

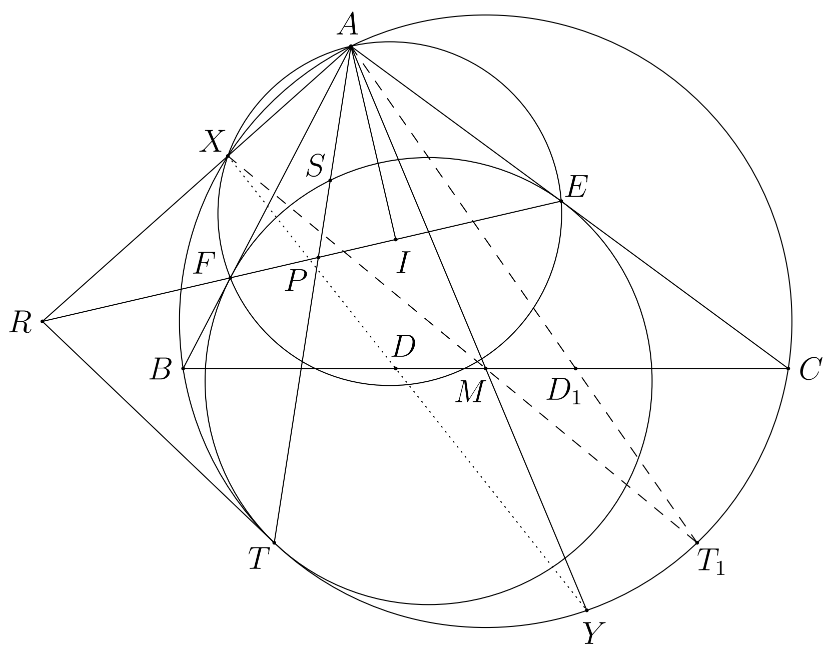
$$\frac{BF}{CE} = \frac{BF}{FI} \cdot \frac{EI}{CE} = \left(\frac{\sin\gamma}{\sin\beta}\right)^2. \tag{5}$$

Together with (1) and (2), we find that

$$\frac{BD'}{CD'} = \frac{AC}{AB} \cdot \left(\frac{\sin\gamma}{\sin\beta}\right)^2 = \frac{\sin 2\beta}{\sin 2\gamma} \cdot \left(\frac{\sin\gamma}{\sin\beta}\right)^2 = \frac{\tan\gamma}{\tan\beta} = \frac{ID/CD}{ID/BD} = \frac{BD}{CD}.$$

This shows $D' = D$ and the result follows.

Solution 2. Let ω_A be the A-mixtilinear incircle of triangle ABC. From the properties of mixtilinear incircles, ω_A touches sides AB and AC at F and E respectively. Suppose ω_A is tangent to Γ at T. Let AM meet Γ again at Y, and let D_1, T_1 be the reflections of D and T with respect to the perpendicular bisector of BC respectively. It is well-known that $\angle BAT = \angle D_1AC$ so that A, D_1, T_1 are collinear.



We then show that X, M, T_1 are collinear. Let R be the radical centre of ω_A, Γ and the circumcircle of triangle AEF. Then R lies on AX, EF and the tangent at T to Γ. Let AT meet ω_A again at S and meet EF at P. Obviously, $SFTE$ is a harmonic quadrilateral. Projecting from T, the pencil $(R, P; F, E)$ is harmonic. We further project the pencil onto Γ from A, so that $XBTC$ is a harmonic quadrilateral. As $TT_1 // BC$, the projection from T_1 onto BC maps T to a point at infinity, and hence maps X to the midpoint of BC, which is M. This shows X, M, T_1 are collinear.

We have two ways to finish the proof.

• **Method 1.** Note that both AY and XT_1 are chords of Γ passing through the midpoint M of the chord BC. By the Butterfly Theorem, XY and AT_1 cut BC at a pair of symmetric points with respect to M, and hence X, D, Y are collinear. The proof is thus complete.

• **Method 2.** Here, we finish the proof without using the Butterfly Theorem. As DTT_1D_1 is an isosceles trapezoid, we have

$$\measuredangle YTD = \measuredangle YTT_1 + \measuredangle T_1TD = \measuredangle YAT_1 + \measuredangle AD_1D = \measuredangle YMD$$

so that D, T, Y, M are concyclic. As X, M, T_1 are collinear, we have

$$\measuredangle AYD = \measuredangle MTD = \measuredangle D_1T_1M = \measuredangle AT_1X = \measuredangle AYX.$$

This shows X, D, Y are collinear.

G3. Let $B = (-1, 0)$ and $C = (1, 0)$ be fixed points on the coordinate plane. A nonempty, bounded subset S of the plane is said to be *nice* if

(i) there is a point T in S such that for every point Q in S, the segment TQ lies entirely in S; and

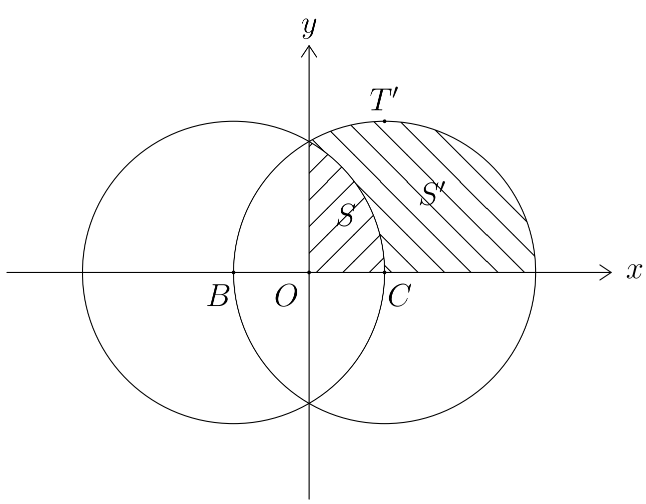
(ii) for any triangle $P_1P_2P_3$, there exists a unique point A in S and a permutation σ of the indices $\{1, 2, 3\}$ for which triangles ABC and $P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$ are similar.

Prove that there exist two distinct nice subsets S and S' of the set $\{(x, y) : x \geqslant 0, y \geqslant 0\}$ such that if $A \in S$ and $A' \in S'$ are the unique choices of points in (ii), then the product $BA \cdot BA'$ is a constant independent of the triangle $P_1P_2P_3$.

Solution. If in the similarity of $\triangle ABC$ and $\triangle P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$, BC corresponds to the longest side of $\triangle P_1P_2P_3$, then we have $BC \geqslant AB \geqslant AC$. The condition $BC \geqslant AB$ is equivalent to $(x + 1)^2 + y^2 \leqslant 4$, while $AB \geqslant AC$ is trivially satisfied for any point in the first quadrant. Then we first define

$$S = \{(x, y) : (x + 1)^2 + y^2 \leqslant 4, x \geqslant 0, y \geqslant 0\}.$$

Note that S is the intersection of a disk and the first quadrant, so it is bounded and convex, and we can choose any $T \in S$ to meet condition (i). For any point A in S, the relation $BC \geqslant AB \geqslant AC$ always holds. Therefore, the point A in (ii) is uniquely determined, while its existence is guaranteed by the above construction.



Next, if in the similarity of $\triangle A'BC$ and $\triangle P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$, BC corresponds to the second longest side of $\triangle P_1P_2P_3$, then we have $A'B \geqslant BC \geqslant A'C$. The two inequalities are equivalent to $(x + 1)^2 + y^2 \geqslant 4$ and $(x - 1)^2 + y^2 \leqslant 4$ respectively. Then we define

$$S' = \{(x, y) : (x + 1)^2 + y^2 \geqslant 4, (x - 1)^2 + y^2 \leqslant 4, x \geqslant 0, y \geqslant 0\}.$$

The boundedness condition is satisfied while (ii) can be argued as in the previous case. For (i), note that S' contains points inside the disk $(x-1)^2+y^2 \leqslant 4$ and outside the disk $(x+1)^2+y^2 \geqslant 4$. This shows we can take $T'=(1,2)$ in (i), which is the topmost point of the circle $(x-1)^2+y^2=4$.

It remains to check that the product $BA \cdot BA'$ is a constant. Suppose we are given a triangle $P_1P_2P_3$ with $P_1P_2 \geqslant P_2P_3 \geqslant P_3P_1$. By the similarity, we have

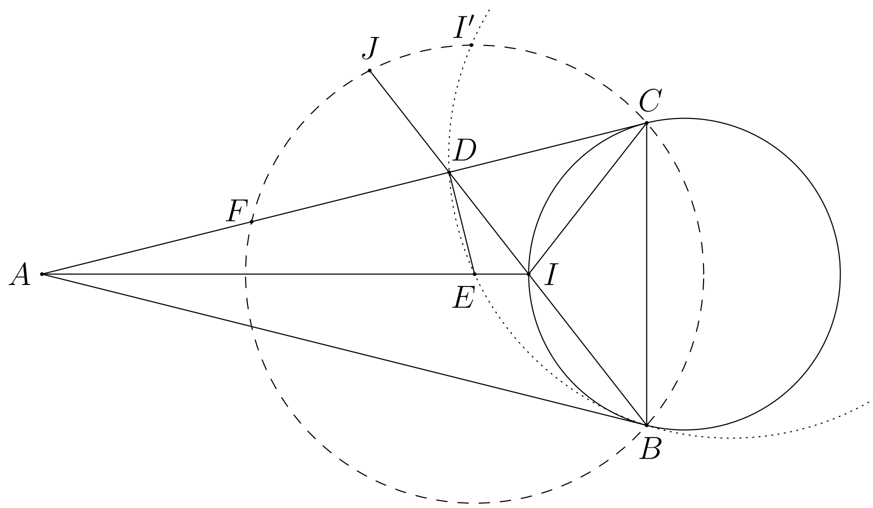
$$BA = BC \cdot \frac{P_2P_3}{P_1P_2} \quad \text{and} \quad BA' = BC \cdot \frac{P_1P_2}{P_2P_3}.$$

Thus $BA \cdot BA' = BC^2 = 4$, which is certainly independent of the triangle $P_1P_2P_3$.

Comment. The original version of this problem includes the condition that the interiors of S and S' are disjoint. We remove this condition since it is hard to define the interior of a point set rigorously.

G4. Let ABC be a triangle with $AB = AC \neq BC$ and let I be its incentre. The line BI meets AC at D, and the line through D perpendicular to AC meets AI at E. Prove that the reflection of I in AC lies on the circumcircle of triangle BDE.

Solution 1.



Let Γ be the circle with centre E passing through B and C. Since $ED \perp AC$, the point F symmetric to C with respect to D lies on Γ. From $\angle DCI = \angle ICB = \angle CBI$, the line DC is a tangent to the circumcircle of triangle IBC. Let J be the symmetric point of I with respect to D. Using directed lengths, from

$$DC \cdot DF = -DC^2 = -DI \cdot DB = DJ \cdot DB,$$

the point J also lies on Γ. Let I' be the reflection of I in AC. Since IJ and CF bisect each other, $CJFI$ is a parallelogram. From $\angle FI'C = \angle CIF = \angle FJC$, we find that I' lies on Γ. This gives $EI' = EB$.

Note that AC is the internal angle bisector of $\angle BDI'$. This shows DE is the external angle bisector of $\angle BDI'$ as $DE \perp AC$. Together with $EI' = EB$, it is well-known that E lies on the circumcircle of triangle BDI'.

Solution 2. Let I' be the reflection of I in AC and let S be the intersection of $I'C$ and AI. Using directed angles, we let $\theta = \measuredangle ACI = \measuredangle ICB = \measuredangle CBI$. We have

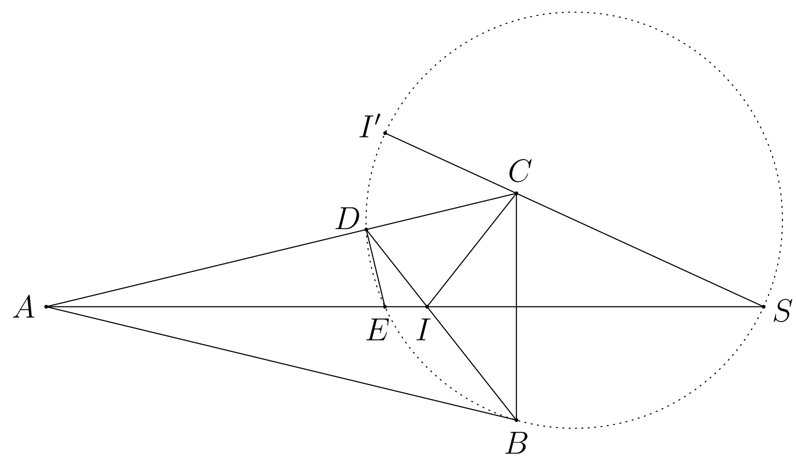
$$\measuredangle I'SE = \measuredangle I'CA + \measuredangle CAI = \theta + \left(\frac{\pi}{2} + 2\theta\right) = 3\theta + \frac{\pi}{2}$$

and

$$\measuredangle I'DE = \measuredangle I'DC + \frac{\pi}{2} = \measuredangle CDI + \frac{\pi}{2} = \measuredangle DCB + \measuredangle CBD + \frac{\pi}{2} = 3\theta + \frac{\pi}{2}.$$

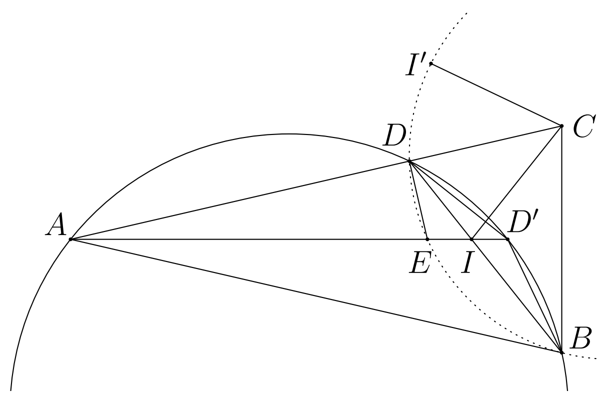
This shows I', D, E, S are concyclic.

Next, we find $\measuredangle I'SB = 2\measuredangle I'SE = 6\theta$ and $\measuredangle I'DB = 2\measuredangle CDI = 6\theta$. Therefore, I', D, B, S are concyclic so that I', D, E, B, S lie on the same circle. The result then follows.



Comment. The point S constructed in Solution 2 may lie on the same side as A of BC. Also, since S lies on the circumcircle of the non-degenerate triangle BDE, we implicitly know that S is not an ideal point. Indeed, one can verify that $I'C$ and AI are parallel if and only if triangle ABC is equilateral.

Solution 3. Let I' be the reflection of I in AC, and let D' be the second intersection of AI and the circumcircle of triangle ABD. Since AD' bisects $\angle BAD$, point D' is the midpoint of the arc BD and $DD' = BD' = CD'$. Obviously, A, E, D' lie on AI in this order.



We find that $\angle ED'D = \angle AD'D = \angle ABD = \angle IBC = \angle ICB$. Next, since D' is the circumcentre of triangle BCD, we have $\angle EDD' = 90^\circ - \angle D'DC = \angle CBD = \angle IBC$. The two relations show that triangles $ED'D$ and ICB are similar. Therefore, we have

$$\frac{BC}{CI'} = \frac{BC}{CI} = \frac{DD'}{D'E} = \frac{BD'}{D'E}.$$

Also, we get

$$\angle BCI' = \angle BCA + \angle ACI' = \angle BCA + \angle ICA = \angle BCA + \angle DBC = \angle BDA = \angle BD'E.$$

These show triangles BCI' and $BD'E$ are similar, and hence triangles BCD' and $BI'E$ are similar. As BCD' is isosceles, we obtain $BE = I'E$.

As DE is the external angle bisector of $\angle BDI'$ and $EI' = EB$, we know that E lies on the circumcircle of triangle BDI'.

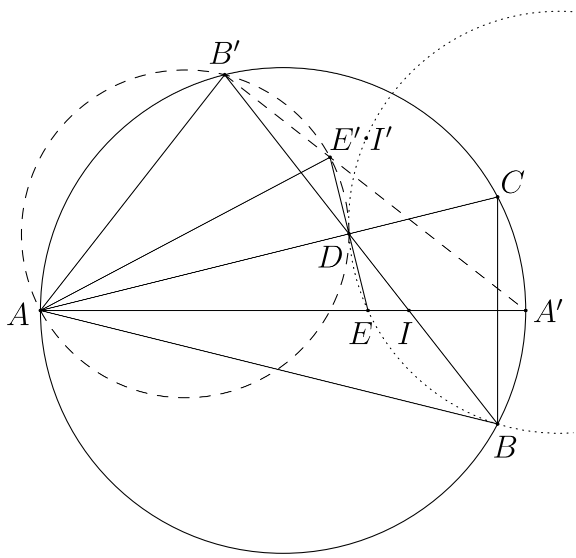
<u>Solution 4</u>. Let AI and BI meet the circumcircle of triangle ABC again at A' and B' respectively, and let E' be the reflection of E in AC. From

$$\angle B'AE' = \angle B'AD - \angle E'AD = \frac{\angle ABC}{2} - \frac{\angle BAC}{2} = 90^\circ - \angle BAC - \frac{\angle ABC}{2}$$
$$= 90^\circ - \angle B'DA = \angle B'DE',$$

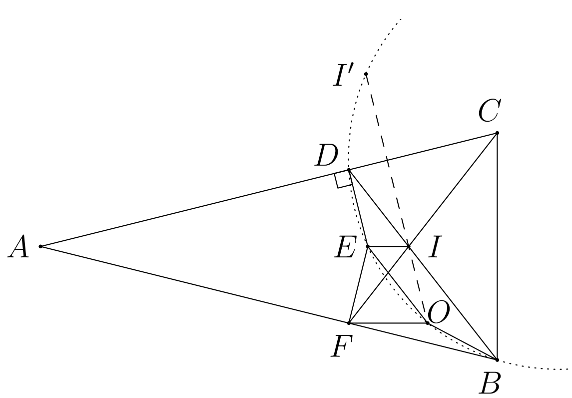
points B', A, D, E' are concyclic. Then

$$\angle DB'E' = \angle DAE' = \frac{\angle BAC}{2} = \angle BAA' = \angle DB'A'$$

and hence B', E', A' are collinear. It is well-known that $A'B'$ is the perpendicular bisector of CI, so that $CE' = IE'$. Let I' be the reflection of I in AC. This implies $BE = CE = I'E$. As DE is the external angle bisector of $\angle BDI'$ and $EI' = EB$, we know that E lies on the circumcircle of triangle BDI'.



<u>Solution 5</u>. Let F be the intersection of CI and AB. Clearly, F and D are symmetric with respect to AI. Let O be the circumcentre of triangle BIF, and let I' be the reflection of I in AC.



From $\angle BFO = 90^\circ - \angle FIB = \frac{1}{2}\angle BAC = \angle BAI$, we get $EI//FO$. Also, from the relation $\angle OIB = 90^\circ - \angle BFI = 90^\circ - \angle CDI = \angle I'ID$, we know that O, I, I' are collinear.

Note that $ED//OI$ since both are perpendicular to AC. Then $\angle FEI = \angle DEI = \angle OIE$. Together with $EI//FO$, the quadrilateral $EFOI$ is an isosceles trapezoid. Therefore, we find that $\angle DIE = \angle FIE = \angle OEI$ so $OE//ID$. Then $DEOI$ is a parallelogram. Hence, we have $DI' = DI = EO$, which shows $DEOI'$ is an isosceles trapezoid. In addition, $ED = OI = OB$ and $OE//BD$ imply $EOBD$ is another isosceles trapezoid. In particular, both $DEOI'$ and $EOBD$ are cyclic. This shows B, D, E, I' are concyclic.

Solution 6. Let I' be the reflection of I in AC. Denote by T and M the projections from I to sides AB and BC respectively. Since BI is the perpendicular bisector of TM, we have

$$DT = DM. \tag{1}$$

Since $\angle ADE = \angle ATI = 90^\circ$ and $\angle DAE = \angle TAI$, we have $\triangle ADE \sim \triangle ATI$. This shows $\frac{AD}{AE} = \frac{AT}{AI} = \frac{AT}{AI'}$. Together with $\angle DAT = 2\angle DAE = \angle EAI'$, this yields $\triangle DAT \sim \triangle EAI'$. In particular, we have

$$\frac{DT}{EI'} = \frac{AT}{AI'} = \frac{AT}{AI}. \tag{2}$$

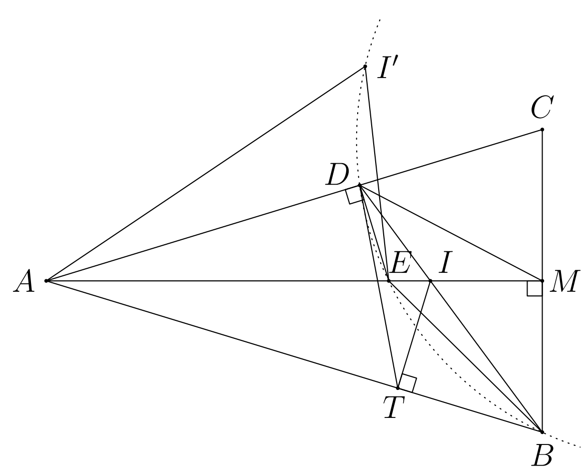
Obviously, the right-angled triangles AMB and ATI are similar. Then we get

$$\frac{AM}{AB} = \frac{AT}{AI}. \tag{3}$$

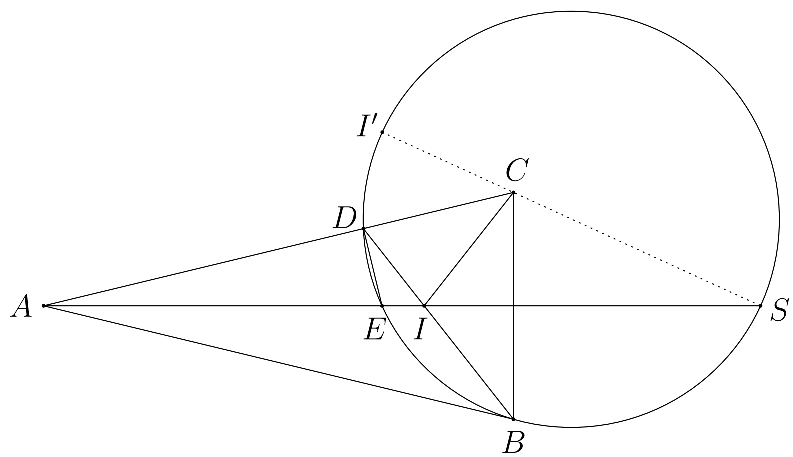
Next, from $\triangle AMB \sim \triangle ATI \sim \triangle ADE$, we have $\frac{AM}{AB} = \frac{AD}{AE}$ so that $\triangle ADM \sim \triangle AEB$. It follows that

$$\frac{DM}{EB} = \frac{AM}{AB}. \tag{4}$$

Combining (1), (2), (3) and (4), we get $EB = EI'$. As DE is the external angle bisector of $\angle BDI'$, we know that E lies on the circumcircle of triangle BDI'.



Comment. A stronger version of this problem is to ask the contestants to prove the reflection of I in AC lies on the circumcircle of triangle BDE if and only if $AB = AC$. Some of the above solutions can be modified to prove the converse statement to the original problem. For example, we borrow some ideas from Solution 2 to establish the converse as follows.



Let I' be the reflection of I in AC and suppose B, E, D, I' lie on a circle Γ. Let AI meet Γ again at S. As DE is the external angle bisector of $\angle BDI'$, we have $EB = EI'$. Using directed angles, we get

$$\measuredangle CI'S = \measuredangle CI'D + \measuredangle DI'S = \measuredangle DIC + \measuredangle DES = \measuredangle DIC + \measuredangle DEA = \measuredangle DIC + \measuredangle DCB = 0.$$

This means I', C, S are collinear. From this we get $\measuredangle BSE = \measuredangle ESI' = \measuredangle ESC$ and hence AS bisects both $\angle BAC$ and $\angle BSC$. Clearly, S and A are distinct points. It follows that $\triangle BAS \cong \triangle CAS$ and thus $AB = AC$.

As in some of the above solutions, an obvious way to prove the stronger version is to establish the following equivalence: $BE = EI'$ if and only if $AB = AC$. In addition to the ideas used in those solutions, one can use trigonometry to express the lengths of BE and EI' in terms of the side lengths of triangle ABC and to establish the equivalence.

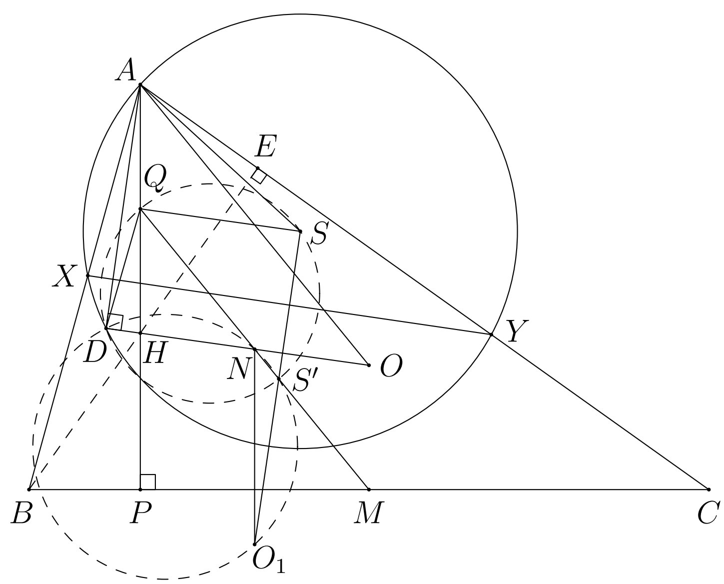
G5. Let D be the foot of perpendicular from A to the Euler line (the line passing through the circumcentre and the orthocentre) of an acute scalene triangle ABC. A circle ω with centre S passes through A and D, and it intersects sides AB and AC at X and Y respectively. Let P be the foot of altitude from A to BC, and let M be the midpoint of BC. Prove that the circumcentre of triangle XSY is equidistant from P and M.

Solution 1. Denote the orthocentre and circumcentre of triangle ABC by H and O respectively. Let Q be the midpoint of AH and N be the nine-point centre of triangle ABC. It is known that Q lies on the nine-point circle of triangle ABC, N is the midpoint of QM and that QM is parallel to AO.

Let the perpendicular from S to XY meet line QM at S'. Let E be the foot of altitude from B to side AC. Since Q and S lie on the perpendicular bisector of AD, using directed angles, we have

$$\begin{aligned}\measuredangle SDQ = \measuredangle QAS = \measuredangle XAS - \measuredangle XAQ &= \left(\frac{\pi}{2} - \measuredangle AYX\right) - \measuredangle BAP = \measuredangle CBA - \measuredangle AYX \\ &= (\measuredangle CBA - \measuredangle ACB) - \measuredangle BCA - \measuredangle AYX = \measuredangle PEM - (\measuredangle BCA + \measuredangle AYX) \\ &= \measuredangle PQM - \angle(BC, XY) = \frac{\pi}{2} - \angle(S'Q, BC) - \angle(BC, XY) = \measuredangle SS'Q.\end{aligned}$$

This shows D, S', S, Q are concyclic.



Let the perpendicular from N to BC intersect line SS' at O_1. (Note that the two lines coincide when S is the midpoint of AO, in which case the result is true since the circumcentre of triangle XSY must lie on this line.) It suffices to show that O_1 is the circumcentre of triangle XSY since N lies on the perpendicular bisector of PM. From

$$\measuredangle DS'O_1 = \measuredangle DQS = \measuredangle SQA = \angle(SQ, QA) = \angle(OD, O_1N) = \measuredangle DNO_1$$

since $SQ//OD$ and $QA//O_1N$, we know that D, O_1, S', N are concyclic. Therefore, we get

$$\measuredangle SDS' = \measuredangle SQS' = \angle(SQ, QS') = \angle(ND, NS') = \measuredangle DNS',$$

so that SD is a tangent to the circle through D, O_1, S', N. Then we have

$$SS' \cdot SO_1 = SD^2 = SX^2. \tag{1}$$

Next, we show that S and S' are symmetric with respect to XY. By the Sine Law, we have

$$\frac{SS'}{\sin \angle SQS'} = \frac{SQ}{\sin \angle SS'Q} = \frac{SQ}{\sin \angle SDQ} = \frac{SQ}{\sin \angle SAQ} = \frac{SA}{\sin \angle SQA}.$$

It follows that

$$SS' = SA \cdot \frac{\sin \angle SQS'}{\sin \angle SQA} = SA \cdot \frac{\sin \angle HOA}{\sin \angle OHA} = SA \cdot \frac{AH}{AO} = SA \cdot 2\cos A,$$

which is twice the distance from S to XY. Note that S and C lie on the same side of the perpendicular bisector of PM if and only if $\angle SAC < \angle OAC$ if and only if $\angle YXA > \angle CBA$. This shows S and O_1 lie on different sides of XY. As S' lies on ray SO_1, it follows that S and S' cannot lie on the same side of XY. Therefore, S and S' are symmetric with respect to XY.

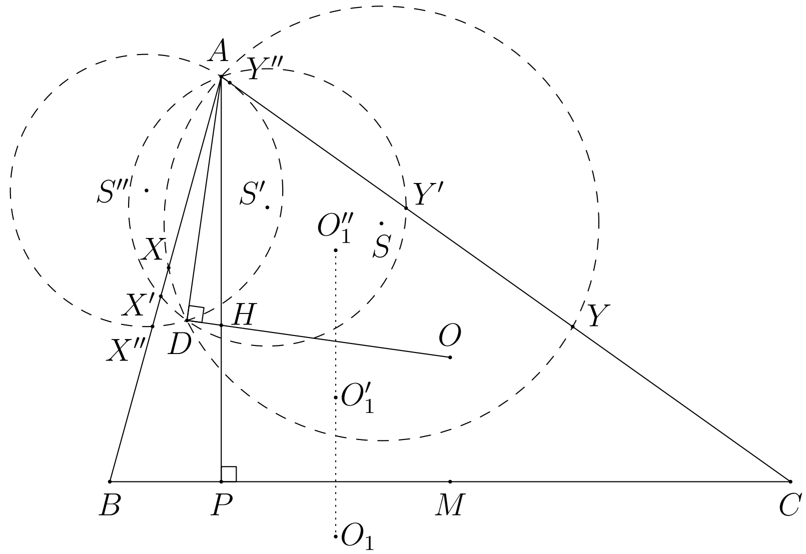
Let d be the diameter of the circumcircle of triangle XSY. As SS' is twice the distance from S to XY and $SX = SY$, we have $SS' = 2\frac{SX^2}{d}$. It follows from (1) that $d = 2SO_1$. As SO_1 is the perpendicular bisector of XY, point O_1 is the circumcentre of triangle XSY.

Solution 2. Denote the orthocentre and circumcentre of triangle ABC by H and O respectively. Let O_1 be the circumcentre of triangle XSY. Consider two other possible positions of S. We name them S' and S'' and define the analogous points $X', Y', O_1', X'', Y''O_1''$ accordingly. Note that S, S', S'' lie on the perpendicular bisector of AD.

As XX' and YY' meet at A and the circumcircles of triangles AXY and $AX'Y'$ meet at D, there is a spiral similarity with centre D mapping XY to $X'Y'$. We find that

$$\measuredangle SXY = \frac{\pi}{2} - \measuredangle YAX = \frac{\pi}{2} - \measuredangle Y'AX' = \measuredangle S'X'Y'$$

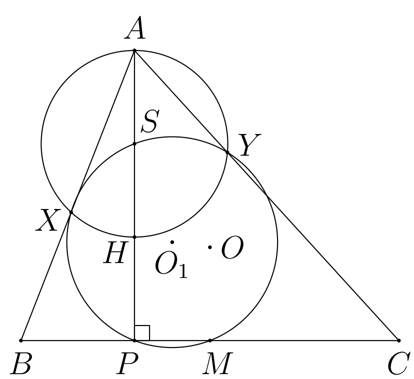
and similarly $\measuredangle SYX = \measuredangle S'Y'X'$. This shows triangles SXY and $S'X'Y'$ are directly similar. Then the spiral similarity with centre D takes points S, X, Y, O_1 to S', X', Y', O_1'. Similarly, there is a spiral similarity with centre D mapping S, X, Y, O_1 to S'', X'', Y'', O_1''. From these, we see that there is a spiral similarity taking the corresponding points S, S', S'' to points O_1, O_1', O_1''. In particular, O_1, O_1', O_1'' are collinear.

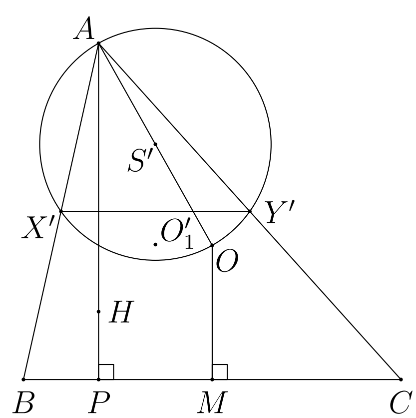


It now suffices to show that O_1 lies on the perpendicular bisector of PM for two special cases.

Firstly, we take S to be the midpoint of AH. Then X and Y are the feet of altitudes from C and B respectively. It is well-known that the circumcircle of triangle XSY is the nine-point circle of triangle ABC. Then O_1 is the nine-point centre and $O_1P = O_1M$. Indeed, P and M also lies on the nine-point circle.

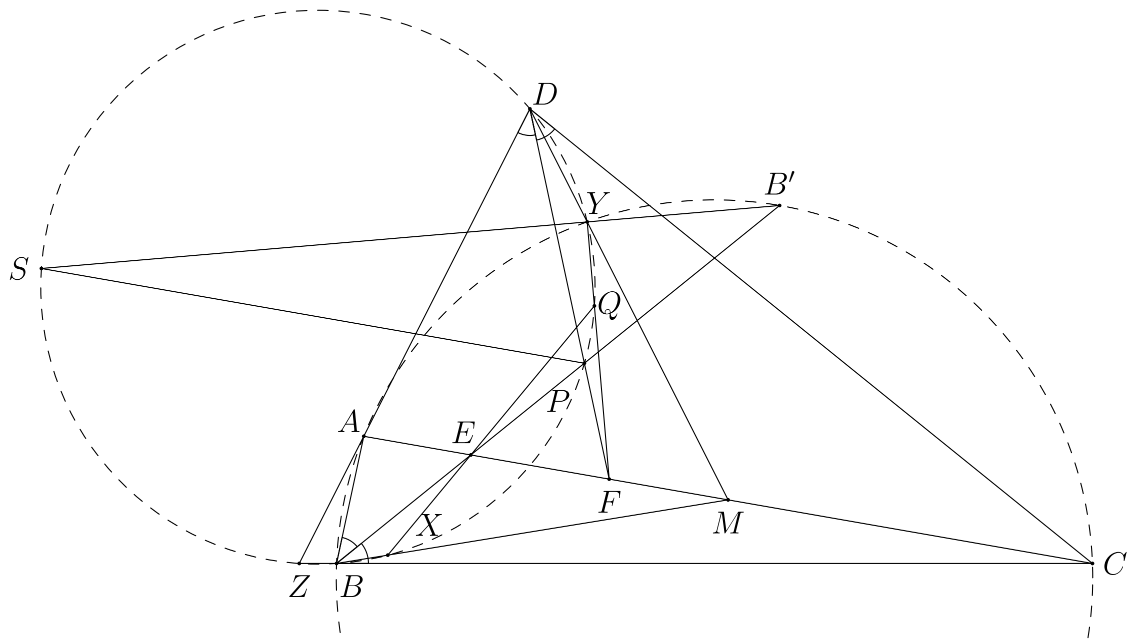
Secondly, we take S' to be the midpoint of AO. Then X' and Y' are the midpoints of AB and AC respectively. Then $X'Y'//BC$. Clearly, S' lies on the perpendicular bisector of PM. This shows the perpendicular bisectors of $X'Y'$ and PM coincide. Hence, we must have $O_1'P = O_1'M$.





G6. Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle ADC < 90°$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at E and F respectively, and meet each other at point P. Let M be the midpoint of AC and let ω be the circumcircle of triangle BPD. Segments BM and DM intersect ω again at X and Y respectively. Denote by Q the intersection point of lines XE and YF. Prove that $PQ \perp AC$.

Solution 1.



Let ω_1 be the circumcircle of triangle ABC. We first prove that Y lies on ω_1. Let Y' be the point on ray MD such that $MY' \cdot MD = MA^2$. Then triangles MAY' and MDA are oppositely similar. Since $MC^2 = MA^2 = MY' \cdot MD$, triangles MCY' and MDC are also oppositely similar. Therefore, using directed angles, we have

$$\measuredangle AY'C = \measuredangle AY'M + \measuredangle MY'C = \measuredangle MAD + \measuredangle DCM = \measuredangle CDA = \measuredangle ABC$$

so that Y' lies on ω_1.

Let Z be the intersection point of lines BC and AD. Since $\measuredangle PDZ = \measuredangle PBC = \measuredangle PBZ$, point Z lies on ω. In addition, from $\measuredangle Y'BZ = \measuredangle Y'BC = \measuredangle Y'AC = \measuredangle Y'AM = \measuredangle Y'DZ$, we also know that Y' lies on ω. Note that $\angle ADC$ is acute implies $MA \neq MD$ so $MY' \neq MD$. Therefore, Y' is the second intersection of DM and ω. Then $Y' = Y$ and hence Y lies on ω_1.

Next, by the Angle Bisector Theorem and the similar triangles, we have

$$\frac{FA}{FC} = \frac{AD}{CD} = \frac{AD}{AM} \cdot \frac{CM}{CD} = \frac{YA}{YM} \cdot \frac{YM}{YC} = \frac{YA}{YC}.$$

Hence, FY is the internal angle bisector of $\angle AYC$.

Let B' be the second intersection of the internal angle bisector of $\angle CBA$ and ω_1. Then B' is the midpoint of arc AC not containing B. Therefore, YB' is the external angle bisector of $\angle AYC$, so that $B'Y \perp FY$.

Denote by l the line through P parallel to AC. Suppose l meets line $B'Y$ at S. From

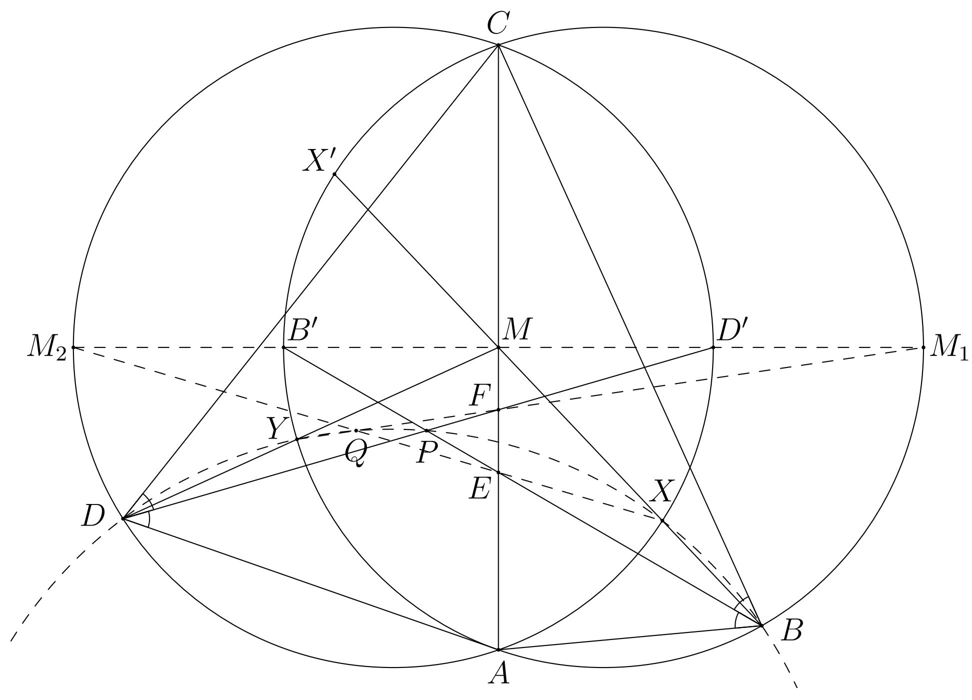
$$\begin{aligned}\measuredangle PSY &= \angle(AC, B'Y) = \measuredangle ACY + \measuredangle CYB' = \measuredangle ACY + \measuredangle CAB' = \measuredangle ACY + \measuredangle B'CA \\ &= \measuredangle B'CY = \measuredangle B'BY = \measuredangle PBY,\end{aligned}$$

the point S lies on ω. Similarly, the line through X perpendicular to XE also passes through the second intersection of l and ω, which is the point S. From $QY \perp YS$ and $QX \perp XS$, point Q lies on ω and QS is a diameter of ω. Therefore, $PQ \perp PS$ so that $PQ \perp AC$.

Solution 2. Denote by ω_1 and ω_2 the circumcircles of triangles ABC and ADC respectively. Since $\angle ABC = \angle ADC$, we know that ω_1 and ω_2 are symmetric with respect to the midpoint M of AC.

Firstly, we show that X lies on ω_2. Let X_1 be the second intersection of ray MB and ω_2 and X' be its symmetric point with respect to M. Then X' lies on ω_1 and $X'AX_1C$ is a parallelogram. Hence, we have

$$\begin{aligned}\measuredangle DX_1B &= \measuredangle DX_1A + \measuredangle AX_1B = \measuredangle DCA + \measuredangle AX_1X' = \measuredangle DCA + \measuredangle CX'X_1 \\ &= \measuredangle DCA + \measuredangle CAB = \angle(CD, AB).\end{aligned}$$



Also, we have

$$\measuredangle DPB = \measuredangle PDC + \angle(CD, AB) + \measuredangle ABP = \angle(CD, AB).$$

These yield $\measuredangle DX_1B = \measuredangle DPB$ and hence X_1 lies on ω. It follows that $X_1 = X$ and X lies on ω_2. Similarly, Y lies on ω_1.

Next, we prove that Q lies on ω. Suppose the perpendicular bisector of AC meet ω_1 at B' and M_1 and meet ω_2 at D' and M_2, so that B, M_1 and D' lie on the same side of AC. Note that B' lies on the angle bisector of $\angle ABC$ and similarly D' lies on DP.

If we denote the area of $W_1W_2W_3$ by $[W_1W_2W_3]$, then

$$\frac{BA \cdot X'A}{BC \cdot X'C} = \frac{\frac{1}{2}BA \cdot X'A \sin \angle BAX'}{\frac{1}{2}BC \cdot X'C \sin \angle BCX'} = \frac{[BAX']}{[BCX']} = \frac{MA}{MC} = 1.$$

As BE is the angle bisector of $\angle ABC$, we have

$$\frac{EA}{EC} = \frac{BA}{BC} = \frac{X'C}{X'A} = \frac{XA}{XC}.$$

Therefore, XE is the angle bisector of $\angle AXC$, so that M_2 lies on the line joining X, E, Q. Analogously, M_1, F, Q, Y are collinear. Thus,

$$\begin{aligned}\measuredangle XQY &= \measuredangle M_2QM_1 = \measuredangle QM_2M_1 + \measuredangle M_2M_1Q = \measuredangle XM_2D' + \measuredangle B'M_1Y \\ &= \measuredangle XDD' + \measuredangle B'BY = \measuredangle XDP + \measuredangle PBY = \measuredangle XBP + \measuredangle PBY = \measuredangle XBY,\end{aligned}$$

which implies Q lies on ω.

Finally, as M_1 and M_2 are symmetric with respect to M, the quadrilateral $X'M_2XM_1$ is a parallelogram. Consequently,

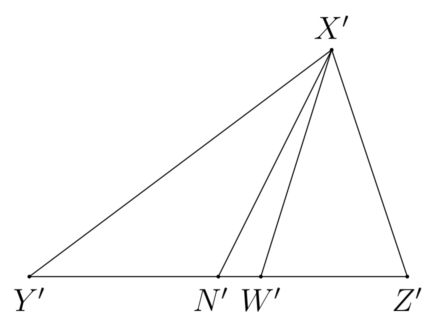
$$\measuredangle XQP = \measuredangle XBP = \measuredangle X'BB' = \measuredangle X'M_1B' = \measuredangle XM_2M_1.$$

This shows $QP // M_2M_1$. As $M_2M_1 \perp AC$, we get $QP \perp AC$.

Solution 3. We first state two results which will be needed in our proof.

• **Claim 1.** In $\triangle X'Y'Z'$ with $X'Y' \neq X'Z'$, let N' be the midpoint of $Y'Z'$ and W' be the foot of internal angle bisector from X'. Then $\tan^2 \measuredangle W'X'Z' = \tan \measuredangle N'X'W' \tan \measuredangle Z'W'X'$.

Proof.



Without loss of generality, assume $X'Y' > X'Z'$. Then W' lies between N' and Z'. The signs of both sides agree so it suffices to establish the relation for ordinary angles. Let $\angle W'X'Z' = \alpha$, $\angle N'X'W' = \beta$ and $\angle Z'W'X' = \gamma$. We have

$$\frac{\sin(\gamma - \alpha)}{\sin(\alpha - \beta)} = \frac{N'X'}{N'Y'} = \frac{N'X'}{N'Z'} = \frac{\sin(\gamma + \alpha)}{\sin(\alpha + \beta)}.$$

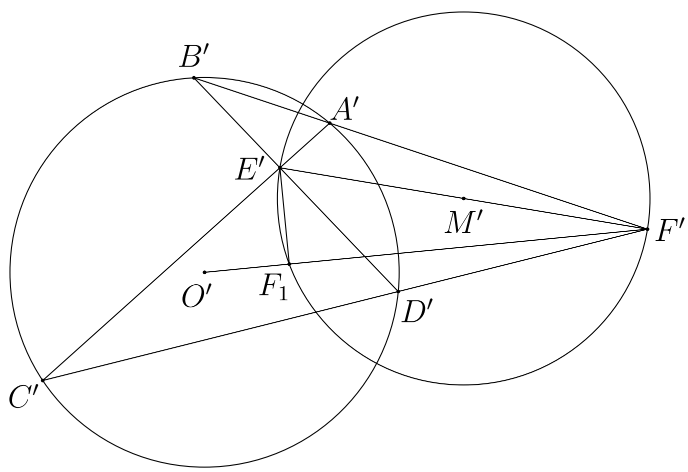
This implies

$$\frac{\tan\gamma-\tan\alpha}{\tan\gamma+\tan\alpha}=\frac{\sin\gamma\cos\alpha-\cos\gamma\sin\alpha}{\sin\gamma\cos\alpha+\cos\gamma\sin\alpha}=\frac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\sin\alpha\cos\beta+\cos\alpha\sin\beta}=\frac{\tan\alpha-\tan\beta}{\tan\alpha+\tan\beta}.$$

Expanding and simplifying, we get the desired result $\tan^2\alpha=\tan\beta\tan\gamma$. □

• **Claim 2.** Let $A'B'C'D'$ be a quadrilateral inscribed in circle Γ. Let diagonals $A'C'$ and $B'D'$ meet at E', and F' be the intersection of lines $A'B'$ and $C'D'$. Let M' be the midpoint of $E'F'$. Then the power of M' with respect to Γ is equal to $(M'E')^2$.

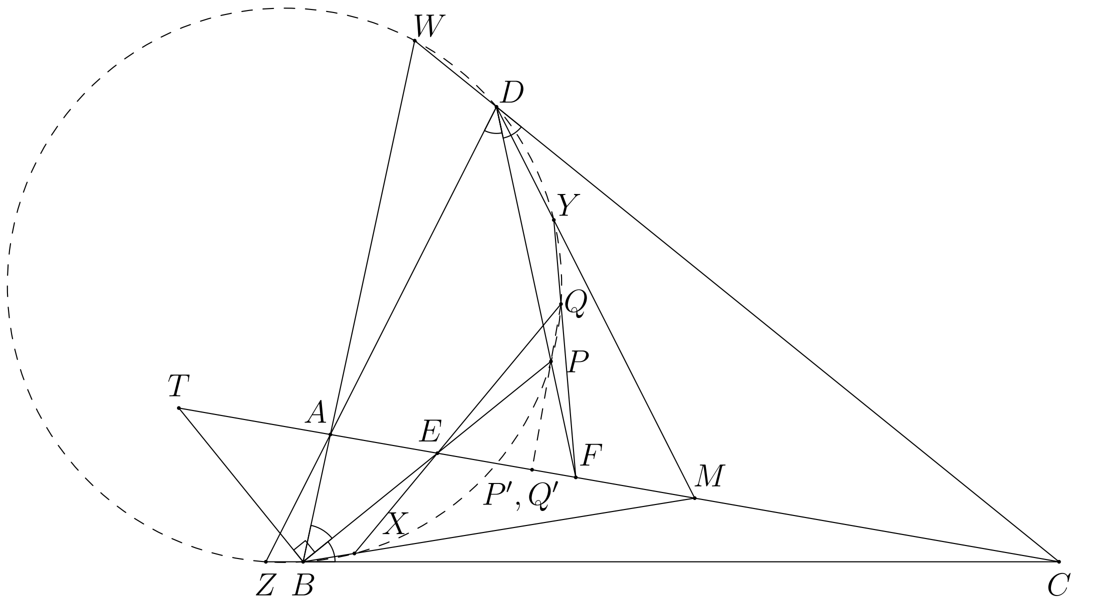
Proof.



Let O' be the centre of Γ and let Γ' be the circle with centre M' passing through E'. Let F_1 be the inversion image of F' with respect to Γ. It is well-known that E' lies on the polar of F' with respect to Γ. This shows $E'F_1 \perp O'F'$ and hence F_1 lies on Γ'. It follows that the inversion image of Γ' with respect to Γ is Γ' itself. This shows Γ' is orthogonal to Γ, and thus the power of M' with respect to Γ is the square of radius of Γ', which is $(M'E')^2$. □

We return to the main problem. Let Z be the intersection of lines AD and BC, and W be the intersection of lines AB and CD. Since $\measuredangle PDZ=\measuredangle PBC=\measuredangle PBZ$, point Z lies on ω. Similarly, W lies on ω. Applying Claim 2 to the cyclic quadrilateral $ZBDW$, we know that the power of M with respect to ω is MA^2. Hence, $MX\cdot MB=MA^2$.

Suppose the line through B perpendicular to BE meets line AC at T. Then BE and BT are the angle bisectors of $\angle CBA$. This shows $(T,E;A,C)$ is harmonic. Thus, we have $ME\cdot MT=MA^2=MX\cdot MB$. It follows that E,T,B,X are concyclic.



The result is trivial for the special case $AD = CD$ since P, Q lie on the perpendicular bisector of AC in that case. Similarly, the case $AB = CB$ is trivial. It remains to consider the general cases where we can apply Claim 1 in the latter part of the proof.

Let the projections from P and Q to AC be P' and Q' respectively. Then $PQ \perp AC$ if and only if $P' = Q'$ if and only if $\frac{EP'}{FP'} = \frac{EQ'}{FQ'}$ in terms of directed lengths. Note that

$$\frac{EP'}{FP'} = \frac{\tan \measuredangle EFP}{\tan \measuredangle FEP} = \frac{\tan \measuredangle AFD}{\tan \measuredangle AEB}.$$

Next, we have $\frac{EQ'}{FQ'} = \frac{\tan \measuredangle EFQ}{\tan \measuredangle FEQ}$ where $\measuredangle FEQ = \measuredangle TEX = \measuredangle TBX = \frac{\pi}{2} + \measuredangle EBM$ and by symmetry $\measuredangle EFQ = \frac{\pi}{2} + \measuredangle FDM$. Combining all these, it suffices to show

$$\frac{\tan \measuredangle AFD}{\tan \measuredangle AEB} = \frac{\tan \measuredangle MBE}{\tan \measuredangle MDF}.$$

We now apply Claim 1 twice to get

$$\tan \measuredangle AFD \tan \measuredangle MDF = \tan^2 \measuredangle FDC = \tan^2 \measuredangle EBA = \tan \measuredangle MBE \tan \measuredangle AEB.$$

The result then follows.

G7. Let I be the incentre of a non-equilateral triangle ABC, I_A be the A-excentre, I'_A be the reflection of I_A in BC, and l_A be the reflection of line AI'_A in AI. Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B.

(a) Prove that P lies on line OI where O is the circumcentre of triangle ABC.

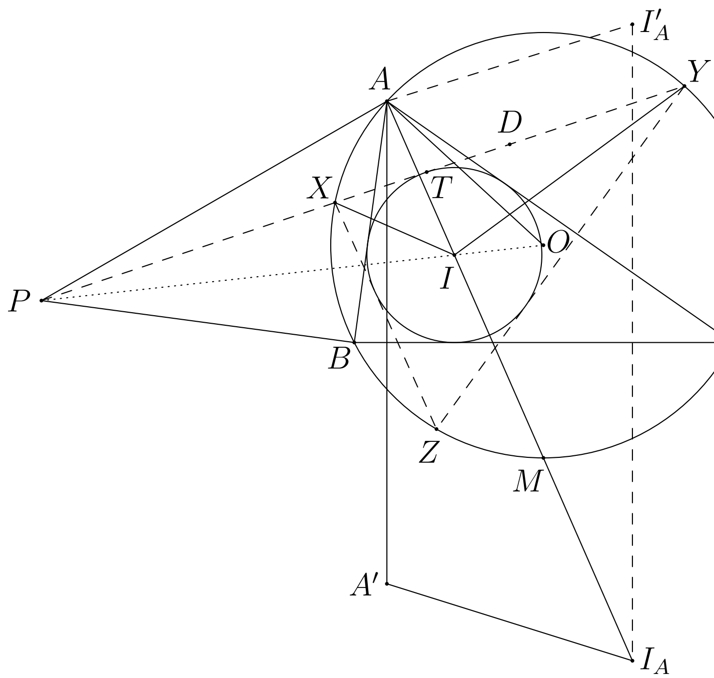
(b) Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y. Show that $\angle XIY = 120°$.

Solution 1.

(a) Let A' be the reflection of A in BC and let M be the second intersection of line AI and the circumcircle Γ of triangle ABC. As triangles ABA' and AOC are isosceles with $\angle ABA' = 2\angle ABC = \angle AOC$, they are similar to each other. Also, triangles ABI_A and AIC are similar. Therefore we have

$$\frac{AA'}{AI_A} = \frac{AA'}{AB} \cdot \frac{AB}{AI_A} = \frac{AC}{AO} \cdot \frac{AI}{AC} = \frac{AI}{AO}.$$

Together with $\angle A'AI_A = \angle IAO$, we find that triangles $AA'I_A$ and AIO are similar.



Denote by P' the intersection of line AP and line OI. Using directed angles, we have

$$\begin{aligned}\measuredangle MAP' &= \measuredangle I'_A A I_A = \measuredangle I'_A AA' - \measuredangle I_A AA' = \measuredangle AA'I_A - \angle(AM, OM) \\ &= \measuredangle AIO - \measuredangle AMO = \measuredangle MOP'.\end{aligned}$$

This shows M, O, A, P' are concyclic.

Denote by R and r the circumradius and inradius of triangle ABC. Then

$$IP' = \frac{IA \cdot IM}{IO} = \frac{IO^2 - R^2}{IO}$$

is independent of A. Hence, BP also meets line OI at the same point P' so that $P' = P$, and P lies on OI.

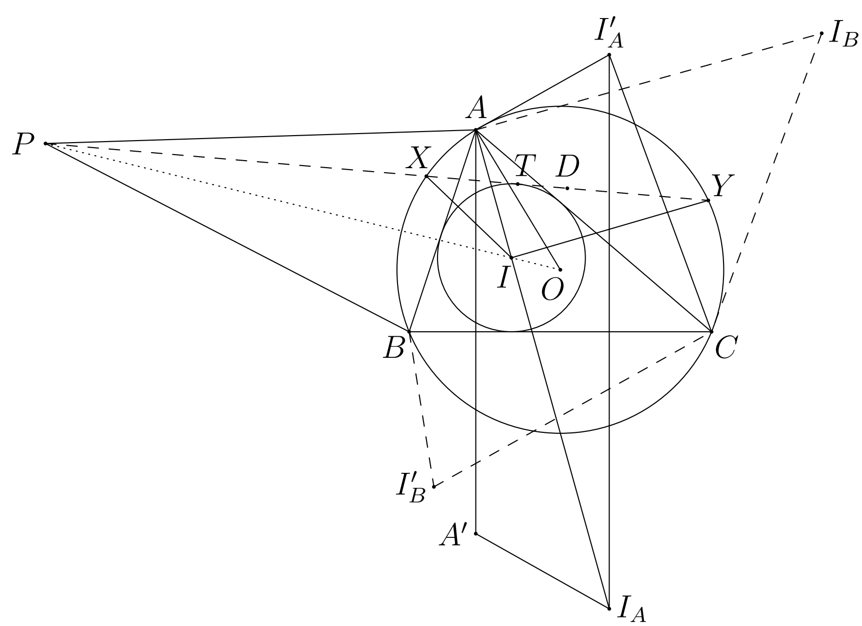
(b) By Poncelet's Porism, the other tangents to the incircle of triangle ABC from X and Y meet at a point Z on Γ. Let T be the touching point of the incircle to XY, and let D be the midpoint of XY. We have

$$OD = IT \cdot \frac{OP}{IP} = r\left(1 + \frac{OI}{IP}\right) = r\left(1 + \frac{OI^2}{OI \cdot IP}\right) = r\left(1 + \frac{R^2 - 2Rr}{R^2 - IO^2}\right)$$
$$= r\left(1 + \frac{R^2 - 2Rr}{2Rr}\right) = \frac{R}{2} = \frac{OX}{2}.$$

This shows $\angle XZY = 60^\circ$ and hence $\angle XIY = 120^\circ$.

Solution 2.

(a) Note that triangles AI_BC and I_ABC are similar since their corresponding interior angles are equal. Therefore, the four triangles AI'_BC, AI_BC, I_ABC and I'_ABC are all similar. From $\triangle AI'_BC \sim \triangle I'_ABC$, we get $\triangle AI'_AC \sim \triangle I'_BBC$. From $\measuredangle ABP = \measuredangle I'_BBC = \measuredangle AI'_AC$ and $\measuredangle BAP = \measuredangle I'_AAC$, the triangles ABP and AI'_AC are directly similar.



Consider the inversion with centre A and radius $\sqrt{AB \cdot AC}$ followed by the reflection in AI. Then B and C are mapped to each other, and I and I_A are mapped to each other.

From the similar triangles obtained, we have $AP \cdot AI'_A = AB \cdot AC$ so that P is mapped to I'_A under the transformation. In addition, line AO is mapped to the altitude from A, and hence O is mapped to the reflection of A in BC, which we call point A'. Note that $AA'I_AI'_A$ is an isosceles trapezoid, which shows it is inscribed in a circle. The preimage of this circle is a straight line, meaning that O, I, P are collinear.

(b) Denote by R and r the circumradius and inradius of triangle ABC. Note that by the above transformation, we have $\triangle APO \sim \triangle AA'I'_A$ and $\triangle AA'I_A \sim \triangle AIO$. Therefore, we find that

$$PO = A'I'_A \cdot \frac{AO}{AI'_A} = AI_A \cdot \frac{AO}{A'I_A} = \frac{AI_A}{A'I_A} \cdot AO = \frac{AO}{IO} \cdot AO.$$

This shows $PO \cdot IO = R^2$, and it follows that P and I are mapped to each other under the inversion with respect to the circumcircle Γ of triangle ABC. Then $PX \cdot PY$, which is the power of P with respect to Γ, equals $PI \cdot PO$. This yields X, I, O, Y are concyclic.

Let T be the touching point of the incircle to XY, and let D be the midpoint of XY. Then

$$OD = IT \cdot \frac{PO}{PI} = r \cdot \frac{PO}{PO - IO} = r \cdot \frac{R^2}{R^2 - IO^2} = r \cdot \frac{R^2}{2Rr} = \frac{R}{2}.$$

This shows $\angle DOX = 60^\circ$ and hence $\angle XIY = \angle XOY = 120^\circ$.

Comment. A simplification of this problem is to ask part (a) only. Note that the question in part (b) implicitly requires P to lie on OI, or otherwise the angle is not uniquely determined as we can find another tangent from P to the incircle.

G8. Let A_1, B_1 and C_1 be points on sides BC, CA and AB of an acute triangle ABC respectively, such that AA_1, BB_1 and CC_1 are the internal angle bisectors of triangle ABC. Let I be the incentre of triangle ABC, and H be the orthocentre of triangle $A_1B_1C_1$. Show that

$$AH + BH + CH \geqslant AI + BI + CI.$$

Solution. Without loss of generality, assume $\alpha = \angle BAC \leqslant \beta = \angle CBA \leqslant \gamma = \angle ACB$. Denote by a, b, c the lengths of BC, CA, AB respectively. We first show that triangle $A_1B_1C_1$ is acute.

Choose points D and E on side BC such that $B_1D // AB$ and B_1E is the internal angle bisector of $\angle BB_1C$. As $\angle B_1DB = 180^\circ - \beta$ is obtuse, we have $BB_1 > B_1D$. Thus,

$$\frac{BE}{EC} = \frac{BB_1}{B_1C} > \frac{DB_1}{B_1C} = \frac{BA}{AC} = \frac{BA_1}{A_1C}.$$

Therefore, $BE > BA_1$ and $\frac{1}{2}\angle BB_1C = \angle BB_1E > \angle BB_1A_1$. Similarly, $\frac{1}{2}\angle BB_1A > \angle BB_1C_1$. It follows that

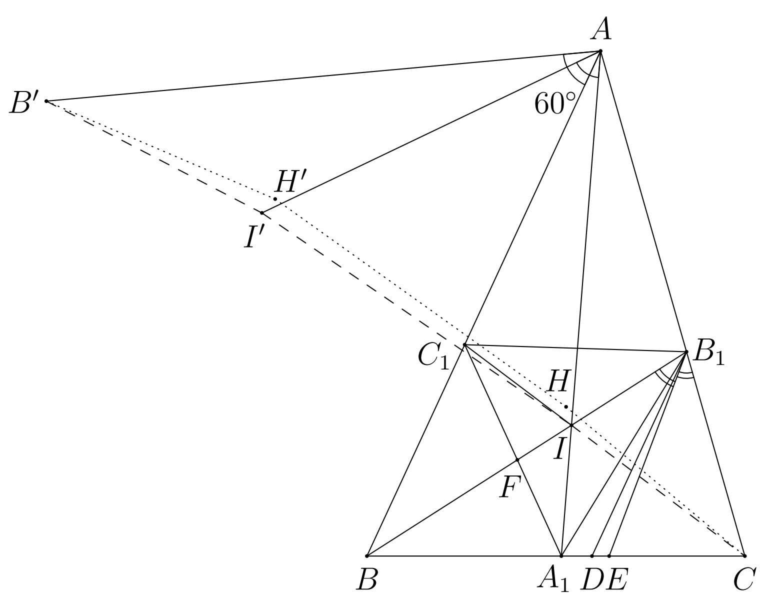
$$\angle A_1B_1C_1 = \angle BB_1A_1 + \angle BB_1C_1 < \frac{1}{2}(\angle BB_1C + \angle BB_1A) = 90^\circ$$

is acute. By symmetry, triangle $A_1B_1C_1$ is acute.

Let BB_1 meet A_1C_1 at F. From $\alpha \leqslant \gamma$, we get $a \leqslant c$, which implies

$$BA_1 = \frac{ca}{b+c} \leqslant \frac{ac}{a+b} = BC_1$$

and hence $\angle BC_1A_1 \leqslant \angle BA_1C_1$. As BF is the internal angle bisector of $\angle A_1BC_1$, this shows $\angle B_1FC_1 = \angle BFA_1 \leqslant 90^\circ$. Hence, H lies on the same side of BB_1 as C_1. This shows H lies inside triangle BB_1C_1. Similarly, from $\alpha \leqslant \beta$ and $\beta \leqslant \gamma$, we know that H lies inside triangles CC_1B_1 and AA_1C_1.



As $\alpha \leqslant \beta \leqslant \gamma$, we have $\alpha \leqslant 60^\circ \leqslant \gamma$. Then $\angle BIC \leqslant 120^\circ \leqslant \angle AIB$. Firstly, suppose $\angle AIC \geqslant 120^\circ$.

Rotate points B, I, H through 60° about A to B', I', H' so that B' and C lie on different sides of AB. Since triangle $AI'I$ is equilateral, we have

$$AI + BI + CI = I'I + B'I' + IC = B'I' + I'I + IC. \tag{1}$$

Similarly,

$$AH + BH + CH = H'H + B'H' + HC = B'H' + H'H + HC. \tag{2}$$

As $\angle AII' = \angle AI'I = 60^\circ$, $\angle AI'B' = \angle AIB \geqslant 120^\circ$ and $\angle AIC \geqslant 120^\circ$, the quadrilateral $B'I'IC$ is convex and lies on the same side of $B'C$ as A.

Next, since H lies inside triangle ACC_1, H lies outside $B'I'IC$. Also, H lying inside triangle ABI implies H' lies inside triangle $AB'I'$. This shows H' lies outside $B'I'IC$ and hence the convex quadrilateral $B'I'IC$ is contained inside the quadrilateral $B'H'HC$. It follows that the perimeter of $B'I'IC$ cannot exceed the perimeter of $B'H'HC$. From (1) and (2), we conclude that

$$AH + BH + CH \geqslant AI + BI + CI.$$

For the case $\angle AIC < 120^\circ$, we can rotate B, I, H through 60° about C to B', I', H' so that B' and A lie on different sides of BC. The proof is analogous to the previous case and we still get the desired inequality.

Number Theory

N1. For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geqslant 2016$, the integer $P(n)$ is positive and

$$S(P(n)) = P(S(n)). \tag{1}$$

Answer.

- $P(x) = c$ where $1 \leqslant c \leqslant 9$ is an integer; or
- $P(x) = x$.

Solution 1. We consider three cases according to the degree of P.

- **Case 1.** $P(x)$ is a constant polynomial.

Let $P(x) = c$ where c is an integer constant. Then (1) becomes $S(c) = c$. This holds if and only if $1 \leqslant c \leqslant 9$.

- **Case 2.** $\deg P = 1$.

We have the following observation. For any positive integers m, n, we have

$$S(m+n) \leqslant S(m) + S(n), \tag{2}$$

and equality holds if and only if there is no carry in the addition $m + n$.

Let $P(x) = ax + b$ for some integers a, b where $a \neq 0$. As $P(n)$ is positive for large n, we must have $a \geqslant 1$. The condition (1) becomes $S(an + b) = aS(n) + b$ for all $n \geqslant 2016$. Setting $n = 2025$ and $n = 2020$ respectively, we get

$$S(2025a + b) - S(2020a + b) = (aS(2025) + b) - (aS(2020) + b) = 5a.$$

On the other hand, (2) implies

$$S(2025a + b) = S((2020a + b) + 5a) \leqslant S(2020a + b) + S(5a).$$

These give $5a \leqslant S(5a)$. As $a \geqslant 1$, this holds only when $a = 1$, in which case (1) reduces to $S(n + b) = S(n) + b$ for all $n \geqslant 2016$. Then we find that

$$S(n + 1 + b) - S(n + b) = (S(n + 1) + b) - (S(n) + b) = S(n + 1) - S(n). \tag{3}$$

If $b > 0$, we choose n such that $n + 1 + b = 10^k$ for some sufficiently large k. Note that all the digits of $n + b$ are 9's, so that the left-hand side of (3) equals $1 - 9k$. As n is a positive integer less than $10^k - 1$, we have $S(n) < 9k$. Therefore, the right-hand side of (3) is at least $1 - (9k - 1) = 2 - 9k$, which is a contradiction.

The case $b < 0$ can be handled similarly by considering $n + 1$ to be a large power of 10. Therefore, we conclude that $P(x) = x$, in which case (1) is trivially satisfied.

- **Case 3.** $\deg P \geqslant 2$.

Suppose the leading term of P is $a_d n^d$ where $a_d \neq 0$. Clearly, we have $a_d > 0$. Consider $n = 10^k - 1$ in (1). We get $S(P(n)) = P(9k)$. Note that $P(n)$ grows asymptotically as fast as n^d, so $S(P(n))$ grows asymptotically as no faster than a constant multiple of k. On the other hand, $P(9k)$ grows asymptotically as fast as k^d. This shows the two sides of the last equation cannot be equal for sufficiently large k since $d \geqslant 2$.

Therefore, we conclude that $P(x) = c$ where $1 \leqslant c \leqslant 9$ is an integer, or $P(x) = x$.

Solution 2. Let $P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0$. Clearly $a_d > 0$. There exists an integer $m \geqslant 1$ such that $|a_i| < 10^m$ for all $0 \leqslant i \leqslant d$. Consider $n = 9 \times 10^k$ for a sufficiently large integer k in (1). If there exists an index $0 \leqslant i \leqslant d-1$ such that $a_i < 0$, then all digits of $P(n)$ in positions from 10^{ik+m+1} to $10^{(i+1)k-1}$ are all 9's. Hence, we have $S(P(n)) \geqslant 9(k-m-1)$. On the other hand, $P(S(n)) = P(9)$ is a fixed constant. Therefore, (1) cannot hold for large k. This shows $a_i \geqslant 0$ for all $0 \leqslant i \leqslant d-1$.

Hence, $P(n)$ is an integer formed by the nonnegative integers $a_d \times 9^d, a_{d-1} \times 9^{d-1}, \ldots, a_0$ by inserting some zeros in between. This yields

$$S(P(n)) = S(a_d \times 9^d) + S(a_{d-1} \times 9^{d-1}) + \cdots + S(a_0).$$

Combining with (1), we have

$$S(a_d \times 9^d) + S(a_{d-1} \times 9^{d-1}) + \cdots + S(a_0) = P(9) = a_d \times 9^d + a_{d-1} \times 9^{d-1} + \cdots + a_0.$$

As $S(m) \leqslant m$ for any positive integer m, with equality when $1 \leqslant m \leqslant 9$, this forces each $a_i \times 9^i$ to be a positive integer between 1 and 9. In particular, this shows $a_i = 0$ for $i \geqslant 2$ and hence $d \leqslant 1$. Also, we have $a_1 \leqslant 1$ and $a_0 \leqslant 9$. If $a_1 = 1$ and $1 \leqslant a_0 \leqslant 9$, we take $n = 10^k + (10 - a_0)$ for sufficiently large k in (1). This yields a contradiction since

$$S(P(n)) = S(10^k + 10) = 2 \neq 11 = P(11 - a_0) = P(S(n)).$$

The zero polynomial is also rejected since $P(n)$ is positive for large n. The remaining candidates are $P(x) = x$ or $P(x) = a_0$ where $1 \leqslant a_0 \leqslant 9$, all of which satisfy (1), and hence are the only solutions.

N2. Let $\tau(n)$ be the number of positive divisors of n. Let $\tau_1(n)$ be the number of positive divisors of n which have remainders 1 when divided by 3. Find all possible integral values of the fraction $\frac{\tau(10n)}{\tau_1(10n)}$.

Answer. All composite numbers together with 2.

Solution. In this solution, we always use p_i to denote primes congruent to 1 mod 3, and use q_j to denote primes congruent to 2 mod 3. When we express a positive integer m using its prime factorization, we also include the special case $m = 1$ by allowing the exponents to be zeros. We first compute $\tau_1(m)$ for a positive integer m.

- **Claim.** Let $m = 3^x p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s} q_1^{b_1} q_2^{b_2} \cdots q_t^{b_t}$ be the prime factorization of m. Then

$$\tau_1(m) = \prod_{i=1}^{s}(a_i+1)\left\lceil \frac{1}{2}\prod_{j=1}^{t}(b_j+1)\right\rceil. \tag{1}$$

Proof. To choose a divisor of m congruent to 1 mod 3, it cannot have the prime divisor 3, while there is no restriction on choosing prime factors congruent to 1 mod 3. Also, we have to choose an even number of prime factors (counted with multiplicity) congruent to 2 mod 3.

If $\prod_{j=1}^{t}(b_j+1)$ is even, then we may assume without loss of generality b_1+1 is even. We can choose the prime factors $q_2, q_3, \ldots, q_t$ freely in $\prod_{j=2}^{t}(b_j+1)$ ways. Then the parity of the number of q_1 is uniquely determined, and hence there are $\frac{1}{2}(b_1+1)$ ways to choose the exponent of q_1. Hence (1) is verified in this case.

If $\prod_{j=1}^{t}(b_j+1)$ is odd, we use induction on t to count the number of choices. When $t = 1$, there are $\lceil \frac{b_1+1}{2} \rceil$ choices for which the exponent is even and $\lfloor \frac{b_1+1}{2} \rfloor$ choices for which the exponent is odd. For the inductive step, we find that there are

$$\left\lceil \frac{1}{2}\prod_{j=1}^{t-1}(b_j+1)\right\rceil \cdot \left\lceil \frac{b_t+1}{2}\right\rceil + \left\lfloor \frac{1}{2}\prod_{j=1}^{t-1}(b_j+1)\right\rfloor \cdot \left\lfloor \frac{b_t+1}{2}\right\rfloor = \left\lceil \frac{1}{2}\prod_{j=1}^{t}(b_j+1)\right\rceil$$

choices with an even number of prime factors and hence $\lfloor \frac{1}{2}\prod_{j=1}^{t}(b_j+1)\rfloor$ choices with an odd number of prime factors. Hence (1) is also true in this case. □

Let $n = 3^x 2^y 5^z p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s} q_1^{b_1} q_2^{b_2} \cdots q_t^{b_t}$. Using the well-known formula for computing the divisor function, we get

$$\tau(10n) = (x+1)(y+2)(z+2)\prod_{i=1}^{s}(a_i+1)\prod_{j=1}^{t}(b_j+1). \tag{2}$$

By the Claim, we have

$$\tau_1(10n) = \prod_{i=1}^{s}(a_i+1)\left\lceil \frac{1}{2}(y+2)(z+2)\prod_{j=1}^{t}(b_j+1)\right\rceil. \tag{3}$$

If $c = (y+2)(z+2)\prod_{j=1}^{t}(b_j+1)$ is even, then (2) and (3) imply

$$\frac{\tau(10n)}{\tau_1(10n)} = 2(x+1).$$

In this case $\frac{\tau(10n)}{\tau_1(10n)}$ can be any even positive integer as x runs through all nonnegative integers.

If c is odd, which means y, z are odd and each b_j is even, then (2) and (3) imply

$$\frac{\tau(10n)}{\tau_1(10n)} = \frac{2(x+1)c}{c+1}. \tag{4}$$

For this to be an integer, we need $c+1$ divides $2(x+1)$ since c and $c+1$ are relatively prime. Let $2(x+1) = k(c+1)$. Then (4) reduces to

$$\frac{\tau(10n)}{\tau_1(10n)} = kc = k(y+2)(z+2)\prod_{j=1}^{t}(b_j+1). \tag{5}$$

Noting that y, z are odd, the integers $y+2$ and $z+2$ are at least 3. This shows the integer in this case must be composite. On the other hand, for any odd composite number ab with $a, b \geqslant 3$, we may simply take $n = 3^{\frac{ab-1}{2}} \cdot 2^{a-2} \cdot 5^{b-2}$ so that $\frac{\tau(10n)}{\tau_1(10n)} = ab$ from (5).

We conclude that the fraction can be any even integer or any odd composite number. Equivalently, it can be 2 or any composite number.

N3. Define $P(n) = n^2 + n + 1$. For any positive integers a and b, the set

$$\{P(a), P(a+1), P(a+2), \ldots, P(a+b)\}$$

is said to be *fragrant* if none of its elements is relatively prime to the product of the other elements. Determine the smallest size of a fragrant set.

Answer. 6.

Solution. We have the following observations.

(i) $(P(n), P(n+1)) = 1$ for any n.

We have $(P(n), P(n+1)) = (n^2+n+1, n^2+3n+3) = (n^2+n+1, 2n+2)$. Noting that n^2+n+1 is odd and $(n^2+n+1, n+1) = (1, n+1) = 1$, the claim follows.

(ii) $(P(n), P(n+2)) = 1$ for $n \not\equiv 2 \pmod 7$ and $(P(n), P(n+2)) = 7$ for $n \equiv 2 \pmod 7$.

From $(2n+7)P(n) - (2n-1)P(n+2) = 14$ and the fact that $P(n)$ is odd, $(P(n), P(n+2))$ must be a divisor of 7. The claim follows by checking $n \equiv 0, 1, \ldots, 6 \pmod 7$ directly.

(iii) $(P(n), P(n+3)) = 1$ for $n \not\equiv 1 \pmod 3$ and $3 | (P(n), P(n+3))$ for $n \equiv 1 \pmod 3$.

From $(n+5)P(n) - (n-1)P(n+3) = 18$ and the fact that $P(n)$ is odd, $(P(n), P(n+3))$ must be a divisor of 9. The claim follows by checking $n \equiv 0, 1, 2 \pmod 3$ directly.

Suppose there exists a fragrant set with at most 5 elements. We may assume it contains exactly 5 elements $P(a), P(a+1), \ldots, P(a+4)$ since the following argument also works with fewer elements. Consider $P(a+2)$. From (i), it is relatively prime to $P(a+1)$ and $P(a+3)$. Without loss of generality, assume $(P(a), P(a+2)) > 1$. From (ii), we have $a \equiv 2 \pmod 7$. The same observation implies $(P(a+1), P(a+3)) = 1$. In order that the set is fragrant, $(P(a), P(a+3))$ and $(P(a+1), P(a+4))$ must both be greater than 1. From (iii), this holds only when both a and $a+1$ are congruent to 1 mod 3, which is a contradiction.

It now suffices to construct a fragrant set of size 6. By the Chinese Remainder Theorem, we can take a positive integer a such that

$$a \equiv 7 \pmod{19}, \quad a+1 \equiv 2 \pmod 7, \quad a+2 \equiv 1 \pmod 3.$$

For example, we may take $a = 197$. From (ii), both $P(a+1)$ and $P(a+3)$ are divisible by 7. From (iii), both $P(a+2)$ and $P(a+5)$ are divisible by 3. One also checks from $19 | P(7) = 57$ and $19 | P(11) = 133$ that $P(a)$ and $P(a+4)$ are divisible by 19. Therefore, the set $\{P(a), P(a+1), \ldots, P(a+5)\}$ is fragrant.

Therefore, the smallest size of a fragrant set is 6.

Comment. "Fragrant Harbour" is the English translation of "Hong Kong".

A stronger version of this problem is to show that there exists a fragrant set of size k for any $k \geqslant 6$. We present a proof here.

For each even positive integer m which is not divisible by 3, since $m^2 + 3 \equiv 3 \pmod 4$, we can find a prime $p_m \equiv 3 \pmod 4$ such that $p_m | m^2 + 3$. Clearly, $p_m > 3$.

If $b = 2t \geqslant 6$, we choose a such that $3|2(a+t)+1$ and $p_m|2(a+t)+1$ for each $1 \leqslant m \leqslant b$ with $m \equiv 2, 4 \pmod 6$. For $0 \leqslant r \leqslant t$ and $3|r$, we have $a+t \pm r \equiv 1 \pmod 3$ so that $3|P(a+t \pm r)$. For $0 \leqslant r \leqslant t$ and $(r, 3) = 1$, we have

$$4P(a+t \pm r) \equiv (-1 \pm 2r)^2 + 2(-1 \pm 2r) + 4 = 4r^2 + 3 \equiv 0 \pmod{p_{2r}}.$$

Hence, $\{P(a), P(a+1), \ldots, P(a+b)\}$ is fragrant.

If $b = 2t+1 \geqslant 7$ (the case $b = 5$ has been done in the original problem), we choose a such that $3|2(a+t)+1$ and $p_m|2(a+t)+1$ for $1 \leqslant m \leqslant b$ with $m \equiv 2, 4 \pmod 6$, and that $a + b \equiv 9 \pmod{13}$. Note that a exists by the Chinese Remainder Theorem since $p_m \neq 13$ for all m. The even case shows that $\{P(a), P(a+1), \ldots, P(a+b-1)\}$ is fragrant. Also, one checks from $13|P(9) = 91$ and $13|P(3) = 13$ that $P(a+b)$ and $P(a+b-6)$ are divisible by 13. The proof is thus complete.

N4. Let n, m, k and l be positive integers with $n \neq 1$ such that $n^k + mn^l + 1$ divides $n^{k+l} - 1$. Prove that

- $m = 1$ and $l = 2k$; or
- $l|k$ and $m = \frac{n^{k-l}-1}{n^l-1}$.

Solution 1. It is given that

$$n^k + mn^l + 1 | n^{k+l} - 1. \tag{1}$$

This implies

$$n^k + mn^l + 1 | (n^{k+l} - 1) + (n^k + mn^l + 1) = n^{k+l} + n^k + mn^l. \tag{2}$$

We have two cases to discuss.

- **Case 1.** $l \geqslant k$.
  Since $(n^k + mn^l + 1, n) = 1$, (2) yields

$$n^k + mn^l + 1 | n^l + mn^{l-k} + 1.$$

In particular, we get $n^k + mn^l + 1 \leqslant n^l + mn^{l-k} + 1$. As $n \geqslant 2$ and $k \geqslant 1$, $(m-1)n^l$ is at least $2(m-1)n^{l-k}$. It follows that the inequality cannot hold when $m \geqslant 2$. For $m = 1$, the above divisibility becomes

$$n^k + n^l + 1 | n^l + n^{l-k} + 1.$$

Note that $n^l + n^{l-k} + 1 < n^l + n^l + 1 < 2(n^k + n^l + 1)$. Thus we must have $n^l + n^{l-k} + 1 = n^k + n^l + 1$ so that $l = 2k$, which gives the first result.

- **Case 2.** $l < k$.
  This time (2) yields

$$n^k + mn^l + 1 | n^k + n^{k-l} + m.$$

In particular, we get $n^k + mn^l + 1 \leqslant n^k + n^{k-l} + m$, which implies

$$m \leqslant \frac{n^{k-l} - 1}{n^l - 1}. \tag{3}$$

On the other hand, from (1) we may let $n^{k+l} - 1 = (n^k + mn^l + 1)t$ for some positive integer t. Obviously, t is less than n^l, which means $t \leqslant n^l - 1$ as it is an integer. Then we have $n^{k+l} - 1 \leqslant (n^k + mn^l + 1)(n^l - 1)$, which is the same as

$$m \geqslant \frac{n^{k-l} - 1}{n^l - 1}. \tag{4}$$

Equations (3) and (4) combine to give $m = \frac{n^{k-l}-1}{n^l-1}$. As this is an integer, we have $l | k - l$. This means $l|k$ and it corresponds to the second result.

Solution 2. As in Solution 1, we begin with equation (2).

- **Case 1.** $l \geqslant k$.
  Then (2) yields

$$n^k + mn^l + 1 | n^l + mn^{l-k} + 1.$$

Since $2(n^k+mn^l+1) > 2mn^l+1 > n^l+mn^{l-k}+1$, it follows that $n^k+mn^l+1 = n^l+mn^{l-k}+1$, that is,

$$m(n^l - n^{l-k}) = n^l - n^k.$$

If $m \geqslant 2$, then $m(n^l - n^{l-k}) \geqslant 2n^l - 2n^{l-k} \geqslant 2n^l - n^l > n^l - n^k$ gives a contradiction. Hence $m = 1$ and $l - k = k$, which means $m = 1$ and $l = 2k$.

- **Case 2.** $l < k$.
  Then (2) yields

$$n^k + mn^l + 1 | n^k + n^{k-l} + m.$$

Since $2(n^k+mn^l+1) > 2n^k+m > n^k+n^{k-l}+m$, it follows that $n^k+mn^l+1 = n^k+n^{k-l}+m$. This gives $m = \frac{n^{k-l}-1}{n^l-1}$. Note that $n^l - 1 | n^{k-l} - 1$ implies $l | k - l$ and hence $l | k$. The proof is thus complete.

Comment. Another version of this problem is as follows: let n, m, k and l be positive integers with $n \neq 1$ such that k and l do not divide each other. Show that $n^k + mn^l + 1$ does not divide $n^{k+l} - 1$.

N5. Let a be a positive integer which is not a square number. Denote by A the set of all positive integers k such that

$$k=\frac{x^2-a}{x^2-y^2} \tag{1}$$

for some integers x and y with $x>\sqrt{a}$. Denote by B the set of all positive integers k such that (1) is satisfied for some integers x and y with $0 \leqslant x<\sqrt{a}$. Prove that $A=B$.

<u>Solution 1</u>. We first prove the following preliminary result.

- **Claim.** For fixed k, let x, y be integers satisfying (1). Then the numbers x_1, y_1 defined by

$$x_1=\frac{1}{2}\left(x-y+\frac{(x-y)^2-4a}{x+y}\right), \quad y_1=\frac{1}{2}\left(x-y-\frac{(x-y)^2-4a}{x+y}\right)$$

are integers and satisfy (1) (with x, y replaced by x_1, y_1 respectively).

Proof. Since $x_1+y_1=x-y$ and

$$x_1=\frac{x^2-xy-2a}{x+y}=-x+\frac{2(x^2-a)}{x+y}=-x+2k(x-y),$$

both x_1 and y_1 are integers. Let $u=x+y$ and $v=x-y$. The relation (1) can be rewritten as

$$u^2-(4k-2)uv+(v^2-4a)=0.$$

By Vieta's Theorem, the number $z=\frac{v^2-4a}{u}$ satisfies

$$v^2-(4k-2)vz+(z^2-4a)=0.$$

Since x_1 and y_1 are defined so that $v=x_1+y_1$ and $z=x_1-y_1$, we can reverse the process and verify (1) for x_1, y_1. $\square$

We first show that $B \subset A$. Take any $k \in B$ so that (1) is satisfied for some integers x, y with $0 \leqslant x<\sqrt{a}$. Clearly, $y \neq 0$ and we may assume y is positive. Since a is not a square, we have $k>1$. Hence, we get $0 \leqslant x<y<\sqrt{a}$. Define

$$x_1=\frac{1}{2}\left|x-y+\frac{(x-y)^2-4a}{x+y}\right|, \quad y_1=\frac{1}{2}\left(x-y-\frac{(x-y)^2-4a}{x+y}\right).$$

By the Claim, x_1, y_1 are integers satisfying (1). Also, we have

$$x_1 \geqslant-\frac{1}{2}\left(x-y+\frac{(x-y)^2-4a}{x+y}\right)=\frac{2a+x(y-x)}{x+y} \geqslant \frac{2a}{x+y}>\sqrt{a}.$$

This implies $k \in A$ and hence $B \subset A$.

Next, we shall show that $A \subset B$. Take any $k \in A$ so that (1) is satisfied for some integers x, y with $x > \sqrt{a}$. Again, we may assume y is positive. Among all such representations of k, we choose the one with smallest $x + y$. Define

$$x_1 = \frac{1}{2}\left|x - y + \frac{(x-y)^2 - 4a}{x+y}\right|, \quad y_1 = \frac{1}{2}\left(x - y - \frac{(x-y)^2 - 4a}{x+y}\right).$$

By the Claim, x_1, y_1 are integers satisfying (1). Since $k > 1$, we get $x > y > \sqrt{a}$. Therefore, we have $y_1 > \frac{4a}{x+y} > 0$ and $\frac{4a}{x+y} < x + y$. It follows that

$$x_1 + y_1 \leqslant \max\left\{x - y, \frac{4a - (x-y)^2}{x+y}\right\} < x + y.$$

If $x_1 > \sqrt{a}$, we get a contradiction due to the minimality of $x + y$. Therefore, we must have $0 \leqslant x_1 < \sqrt{a}$, which means $k \in B$ so that $A \subset B$.

The two subset relations combine to give $A = B$.

Solution 2. The relation (1) is equivalent to

$$ky^2 - (k-1)x^2 = a. \tag{2}$$

Motivated by Pell's Equation, we prove the following, which is essentially the same as the Claim in Solution 1.

• **Claim.** If (x_0, y_0) is a solution to (2), then $((2k-1)x_0 \pm 2ky_0, (2k-1)y_0 \pm 2(k-1)x_0)$ is also a solution to (2).

Proof. We check directly that

$$\begin{aligned}
&k((2k-1)y_0 \pm 2(k-1)x_0)^2 - (k-1)((2k-1)x_0 \pm 2ky_0)^2 \\
&= (k(2k-1)^2 - (k-1)(2k)^2)y_0^2 + (k(2(k-1))^2 - (k-1)(2k-1)^2)x_0^2 \\
&= ky_0^2 - (k-1)x_0^2 = a.
\end{aligned}$$

□

If (2) is satisfied for some $0 \leqslant x < \sqrt{a}$ and nonnegative integer y, then clearly (1) implies $y > x$. Also, we have $k > 1$ since a is not a square number. By the Claim, consider another solution to (2) defined by

$$x_1 = (2k-1)x + 2ky, \quad y_1 = (2k-1)y + 2(k-1)x.$$

It satisfies $x_1 \geqslant (2k-1)x + 2k(x+1) = (4k-1)x + 2k > x$. Then we can replace the old solution by a new one which has a larger value in x. After a finite number of replacements, we must get a solution with $x > \sqrt{a}$. This shows $B \subset A$.

If (2) is satisfied for some $x > \sqrt{a}$ and nonnegative integer y, by the Claim we consider another solution to (2) defined by

$$x_1 = |(2k-1)x - 2ky|, \quad y_1 = (2k-1)y - 2(k-1)x.$$

From (2), we get $\sqrt{k}y > \sqrt{k-1}x$. This implies $ky > \sqrt{k(k-1)}x > (k-1)x$ and hence $(2k-1)x - 2ky < x$. On the other hand, the relation (1) implies $x > y$. Then it is clear that $(2k-1)x - 2ky > -x$. These combine to give $x_1 < x$, which means we have found a solution to (2) with x having a smaller absolute value. After a finite number of steps, we shall obtain a solution with $0 \leqslant x < \sqrt{a}$. This shows $A \subset B$.

The desired result follows from $B \subset A$ and $A \subset B$.

Solution 3. It suffices to show $A \cup B$ is a subset of $A \cap B$. We take any $k \in A \cup B$, which means there exist integers x, y satisfying (1). Since a is not a square, it follows that $k \neq 1$. As in Solution 2, the result follows readily once we have proved the existence of a solution (x_1, y_1) to (1) with $|x_1| > |x|$, and, in case of $x > \sqrt{a}$, another solution (x_2, y_2) with $|x_2| < |x|$.

Without loss of generality, assume $x, y \geqslant 0$. Let $u = x + y$ and $v = x - y$. Then $u \geqslant v$ and (1) becomes

$$k = \frac{(u+v)^2 - 4a}{4uv}. \tag{3}$$

This is the same as

$$v^2 + (2u - 4ku)v + u^2 - 4a = 0.$$

Let $v_1 = 4ku - 2u - v$. Then $u + v_1 = 4ku - u - v \geqslant 8u - u - v > u + v$. By Vieta's Theorem, v_1 satisfies

$$v_1^2 + (2u - 4ku)v_1 + u^2 - 4a = 0.$$

This gives $k = \frac{(u+v_1)^2-4a}{4uv_1}$. As k is an integer, $u + v_1$ must be even. Therefore, $x_1 = \frac{u+v_1}{2}$ and $y_1 = \frac{v_1-u}{2}$ are integers. By reversing the process, we can see that (x_1, y_1) is a solution to (1), with $x_1 = \frac{u+v_1}{2} > \frac{u+v}{2} = x \geqslant 0$. This completes the first half of the proof.

Suppose $x > \sqrt{a}$. Then $u + v > 2\sqrt{a}$ and (3) can be rewritten as

$$u^2 + (2v - 4kv)u + v^2 - 4a = 0.$$

Let $u_2 = 4kv - 2v - u$. By Vieta's Theorem, we have $uu_2 = v^2 - 4a$ and

$$u_2^2 + (2v - 4kv)u_2 + v^2 - 4a = 0. \tag{4}$$

By $u > 0$, $u + v > 2\sqrt{a}$ and (3), we have $v > 0$. If $u_2 \geqslant 0$, then $vu_2 \leqslant uu_2 = v^2 - 4a < v^2$. This shows $u_2 < v \leqslant u$ and $0 < u_2 + v < u + v$. If $u_2 < 0$, then $(u_2 + v) + (u + v) = 4kv > 0$ and $u_2 + v < u + v$ imply $|u_2 + v| < u + v$. In any case, since $u_2 + v$ is even from (4), we can define $x_2 = \frac{u_2+v}{2}$ and $y_2 = \frac{u_2-v}{2}$ so that (1) is satisfied with $|x_2| < x$, as desired. The proof is thus complete.

N6. Denote by $\mathbb{N}$ the set of all positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers m and n, the integer $f(m) + f(n) - mn$ is nonzero and divides $mf(m) + nf(n)$.

Answer. $f(n) = n^2$ for any $n \in \mathbb{N}$.

Solution. It is given that

$$f(m) + f(n) - mn | mf(m) + nf(n). \tag{1}$$

Taking $m = n = 1$ in (1), we have $2f(1) - 1 | 2f(1)$. Then $2f(1) - 1 | 2f(1) - (2f(1) - 1) = 1$ and hence $f(1) = 1$.

Let $p \geqslant 7$ be a prime. Taking $m = p$ and $n = 1$ in (1), we have $f(p) - p + 1 | pf(p) + 1$ and hence

$$f(p) - p + 1 | pf(p) + 1 - p(f(p) - p + 1) = p^2 - p + 1.$$

If $f(p) - p + 1 = p^2 - p + 1$, then $f(p) = p^2$. If $f(p) - p + 1 \neq p^2 - p + 1$, as $p^2 - p + 1$ is an odd positive integer, we have $p^2 - p + 1 \geqslant 3(f(p) - p + 1)$, that is,

$$f(p) \leqslant \frac{1}{3}(p^2 + 2p - 2). \tag{2}$$

Taking $m = n = p$ in (1), we have $2f(p) - p^2 | 2pf(p)$. This implies

$$2f(p) - p^2 | 2pf(p) - p(2f(p) - p^2) = p^3.$$

By (2) and $f(p) \geqslant 1$, we get

$$-p^2 < 2f(p) - p^2 \leqslant \frac{2}{3}(p^2 + 2p - 2) - p^2 < -p$$

since $p \geqslant 7$. This contradicts the fact that $2f(p) - p^2$ is a factor of p^3. Thus we have proved that $f(p) = p^2$ for all primes $p \geqslant 7$.

Let n be a fixed positive integer. Choose a sufficiently large prime p. Consider $m = p$ in (1). We obtain

$$f(p) + f(n) - pn | pf(p) + nf(n) - n(f(p) + f(n) - pn) = pf(p) - nf(p) + pn^2.$$

As $f(p) = p^2$, this implies $p^2 - pn + f(n) | p(p^2 - pn + n^2)$. As p is sufficiently large and n is fixed, p cannot divide $f(n)$, and so $(p, p^2 - pn + f(n)) = 1$. It follows that $p^2 - pn + f(n) | p^2 - pn + n^2$ and hence

$$p^2 - pn + f(n) | (p^2 - pn + n^2) - (p^2 - pn + f(n)) = n^2 - f(n).$$

Note that $n^2 - f(n)$ is fixed while $p^2 - pn + f(n)$ is chosen to be sufficiently large. Therefore, we must have $n^2 - f(n) = 0$ so that $f(n) = n^2$ for any positive integer n.

Finally, we check that when $f(n) = n^2$ for any positive integer n, then

$$f(m) + f(n) - mn = m^2 + n^2 - mn$$

and

$$mf(m) + nf(n) = m^3 + n^3 = (m + n)(m^2 + n^2 - mn).$$

The latter expression is divisible by the former for any positive integers m, n. This shows $f(n) = n^2$ is the only solution.

N7. Let n be an odd positive integer. In the Cartesian plane, a cyclic polygon P with area S is chosen. All its vertices have integral coordinates, and the squares of its side lengths are all divisible by n. Prove that $2S$ is an integer divisible by n.

Solution. Let $P = A_1A_2 \dots A_k$ and let $A_{k+i} = A_i$ for $i \geqslant 1$. By the Shoelace Formula, the area of any convex polygon with integral coordinates is half an integer. Therefore, $2S$ is an integer. We shall prove by induction on $k \geqslant 3$ that $2S$ is divisible by n. Clearly, it suffices to consider $n = p^t$ where p is an odd prime and $t \geqslant 1$.

For the base case $k = 3$, let the side lengths of P be $\sqrt{na}, \sqrt{nb}, \sqrt{nc}$ where a, b, c are positive integers. By Heron's Formula,

$$16S^2 = n^2(2ab + 2bc + 2ca - a^2 - b^2 - c^2).$$

This shows $16S^2$ is divisible by n^2. Since n is odd, $2S$ is divisible by n.

Assume $k \geqslant 4$. If the square of length of one of the diagonals is divisible by n, then that diagonal divides P into two smaller polygons, to which the induction hypothesis applies. Hence we may assume that none of the squares of diagonal lengths is divisible by n. As usual, we denote by $\nu_p(r)$ the exponent of p in the prime decomposition of r. We claim the following.

- **Claim.** $\nu_p(A_1A_m^2) > \nu_p(A_1A_{m+1}^2)$ for $2 \leqslant m \leqslant k-1$.

Proof. The case $m = 2$ is obvious since $\nu_p(A_1A_2^2) \geqslant p^t > \nu_p(A_1A_3^2)$ by the condition and the above assumption.

Suppose $\nu_p(A_1A_2^2) > \nu_p(A_1A_3^2) > \cdots > \nu_p(A_1A_m^2)$ where $3 \leqslant m \leqslant k-1$. For the induction step, we apply Ptolemy's Theorem to the cyclic quadrilateral $A_1A_{m-1}A_mA_{m+1}$ to get

$$A_1A_{m+1} \times A_{m-1}A_m + A_1A_{m-1} \times A_mA_{m+1} = A_1A_m \times A_{m-1}A_{m+1},$$

which can be rewritten as

$$\begin{aligned} A_1A_{m+1}^2 \times A_{m-1}A_m^2 = {} & A_1A_{m-1}^2 \times A_mA_{m+1}^2 + A_1A_m^2 \times A_{m-1}A_{m+1}^2 \\ & - 2A_1A_{m-1} \times A_mA_{m+1} \times A_1A_m \times A_{m-1}A_{m+1}. \end{aligned} \tag{1}$$

From this, $2A_1A_{m-1} \times A_mA_{m+1} \times A_1A_m \times A_{m-1}A_{m+1}$ is an integer. We consider the component of p of each term in (1). By the inductive hypothesis, we have $\nu_p(A_1A_{m-1}^2) > \nu_p(A_1A_m^2)$. Also, we have $\nu_p(A_mA_{m+1}^2) \geqslant p^t > \nu_p(A_{m-1}A_{m+1}^2)$. These give

$$\nu_p(A_1A_{m-1}^2 \times A_mA_{m+1}^2) > \nu_p(A_1A_m^2 \times A_{m-1}A_{m+1}^2). \tag{2}$$

Next, we have $\nu_p(4A_1A_{m-1}^2 \times A_mA_{m+1}^2 \times A_1A_m^2 \times A_{m-1}A_{m+1}^2) = \nu_p(A_1A_{m-1}^2 \times A_mA_{m+1}^2) + \nu_p(A_1A_m^2 \times A_{m-1}A_{m+1}^2) > 2\nu_p(A_1A_m^2 \times A_{m-1}A_{m+1}^2)$ from (2). This implies

$$\nu_p(2A_1A_{m-1} \times A_mA_{m+1} \times A_1A_m \times A_{m-1}A_{m+1}) > \nu_p(A_1A_m^2 \times A_{m-1}A_{m+1}^2). \tag{3}$$

Combining (1), (2) and (3), we conclude that

$$\nu_p(A_1A_{m+1}^2 \times A_{m-1}A_m^2) = \nu_p(A_1A_m^2 \times A_{m-1}A_{m+1}^2).$$

By $\nu_p(A_{m-1}A_m^2) \geqslant p^t > \nu_p(A_{m-1}A_{m+1}^2)$, we get $\nu_p(A_1A_{m+1}^2) < \nu_p(A_1A_m^2)$. The Claim follows by induction. $\square$

From the Claim, we get a chain of inequalities

$$p^t > \nu_p(A_1A_3^2) > \nu_p(A_1A_4^2) > \cdots > \nu_p(A_1A_k^2) \geqslant p^t,$$

which yields a contradiction. Therefore, we can show by induction that $2S$ is divisible by n.

Comment. The condition that P is cyclic is crucial. As a counterexample, consider the rhombus with vertices $(0,3),(4,0),(0,-3),(-4,0)$. Each of its squares of side lengths is divisible by 5, while $2S = 48$ is not.

The proposer also gives a proof for the case n is even. One just needs an extra technical step for the case $p = 2$.

N8. Find all polynomials $P(x)$ of odd degree d and with integer coefficients satisfying the following property: for each positive integer n, there exist n positive integers $x_1, x_2, \ldots, x_n$ such that $\frac{1}{2} < \frac{P(x_i)}{P(x_j)} < 2$ and $\frac{P(x_i)}{P(x_j)}$ is the d-th power of a rational number for every pair of indices i and j with $1 \leqslant i, j \leqslant n$.

Answer. $P(x) = a(rx+s)^d$ where a, r, s are integers with $a \neq 0$, $r \geqslant 1$ and $(r,s) = 1$.

Solution. Let $P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0$. Consider the substitution $y = da_d x + a_{d-1}$. By defining $Q(y) = P(x)$, we find that Q is a polynomial with rational coefficients without the term y^{d-1}. Let $Q(y) = b_d y^d + b_{d-2}y^{d-2} + b_{d-3}y^{d-3} + \cdots + b_0$ and $B = \max_{0 \leqslant i \leqslant d}\{|b_i|\}$ (where $b_{d-1} = 0$).

The condition shows that for each $n \geqslant 1$, there exist integers $y_1, y_2, \ldots, y_n$ such that $\frac{1}{2} < \frac{Q(y_i)}{Q(y_j)} < 2$ and $\frac{Q(y_i)}{Q(y_j)}$ is the d-th power of a rational number for $1 \leqslant i, j \leqslant n$. Since n can be arbitrarily large, we may assume all x_i's and hence y_i's are integers larger than some absolute constant in the following.

By Dirichlet's Theorem, since d is odd, we can find a sufficiently large prime p such that $p \equiv 2 \pmod{d}$. In particular, we have $(p-1, d) = 1$. For this fixed p, we choose n to be sufficiently large. Then by the Pigeonhole Principle, there must be $d+1$ of $y_1, y_2, \ldots, y_n$ which are congruent mod p. Without loss of generality, assume $y_i \equiv y_j \pmod{p}$ for $1 \leqslant i, j \leqslant d+1$. We shall establish the following.

- **Claim.** $\frac{Q(y_i)}{Q(y_1)} = \frac{y_i^d}{y_1^d}$ for $2 \leqslant i \leqslant d+1$.

Proof. Let $\frac{Q(y_i)}{Q(y_1)} = \frac{l^d}{m^d}$ where $(l,m) = 1$ and $l, m > 0$. This can be rewritten in the expanded form

$$b_d(m^d y_i^d - l^d y_1^d) = -\sum_{j=0}^{d-2} b_j (m^d y_i^j - l^d y_1^j). \tag{1}$$

Let c be the common denominator of Q, so that $cQ(k)$ is an integer for any integer k. Note that c depends only on P and so we may assume $(p,c) = 1$. Then $y_1 \equiv y_i \pmod{p}$ implies $cQ(y_1) \equiv cQ(y_i) \pmod{p}$.

- **Case 1.** $p | cQ(y_1)$.

In this case, there is a cancellation of p in the numerator and denominator of $\frac{cQ(y_i)}{cQ(y_1)}$, so that $m^d \leqslant p^{-1}|cQ(y_1)|$. Noting $|Q(y_1)| < 2By_1^d$ as y_1 is large, we get

$$m \leqslant p^{-\frac{1}{d}}(2cB)^{\frac{1}{d}} y_1. \tag{2}$$

For large y_1 and y_i, the relation $\frac{1}{2} < \frac{Q(y_i)}{Q(y_1)} < 2$ implies

$$\frac{1}{3} < \frac{y_i^d}{y_1^d} < 3. \tag{3}$$

We also have

$$\frac{1}{2} < \frac{l^d}{m^d} < 2. \tag{4}$$

Now, the left-hand side of (1) is

$$b_d(my_i - ly_1)(m^{d-1}y_i^{d-1} + m^{d-2}y_i^{d-2}ly_1 + \cdots + l^{d-1}y_1^{d-1}).$$

Suppose on the contrary that $my_i - ly_1 \neq 0$. Then the absolute value of the above expression is at least $|b_d|m^{d-1}y_i^{d-1}$. On the other hand, the absolute value of the right-hand side of (1) is at most

$$\begin{aligned}\sum_{j=0}^{d-2} B(m^d y_i^j + l^d y_1^j) &\leqslant (d-1)B(m^d y_i^{d-2} + l^d y_1^{d-2}) \\ &\leqslant (d-1)B(7m^d y_i^{d-2}) \\ &\leqslant 7(d-1)B(p^{-\frac{1}{d}}(2cB)^{\frac{1}{d}}y_1)m^{d-1}y_i^{d-2} \\ &\leqslant 21(d-1)Bp^{-\frac{1}{d}}(2cB)^{\frac{1}{d}}m^{d-1}y_i^{d-1}\end{aligned}$$

by using successively (3), (4), (2) and again (3). This shows

$$|b_d|m^{d-1}y_i^{d-1} \leqslant 21(d-1)Bp^{-\frac{1}{d}}(2cB)^{\frac{1}{d}}m^{d-1}y_i^{d-1},$$

which is a contradiction for large p as b_d, B, c, d depend only on the polynomial P. Therefore, we have $my_i - ly_1 = 0$ in this case.

- **Case 2.** $(p, cQ(y_1)) = 1$.

From $cQ(y_1) \equiv cQ(y_i) \pmod{p}$, we have $l^d \equiv m^d \pmod{p}$. Since $(p-1, d) = 1$, we use Fermat Little Theorem to conclude $l \equiv m \pmod{p}$. Then $p | my_i - ly_1$. Suppose on the contrary that $my_i - ly_1 \neq 0$. Then the left-hand side of (1) has absolute value at least $|b_d|pm^{d-1}y_i^{d-1}$. Similar to Case 1, the right-hand side of (1) has absolute value at most

$$21(d-1)B(2cB)^{\frac{1}{d}}m^{d-1}y_i^{d-1},$$

which must be smaller than $|b_d|pm^{d-1}y_i^{d-1}$ for large p. Again this yields a contradiction and hence $my_i - ly_1 = 0$.

In both cases, we find that $\frac{Q(y_i)}{Q(y_1)} = \frac{l^d}{m^d} = \frac{y_i^d}{y_1^d}$. □

From the Claim, the polynomial $Q(y_1)y^d - y_1^d Q(y)$ has roots $y = y_1, y_2, \ldots, y_{d+1}$. Since its degree is at most d, this must be the zero polynomial. Hence, $Q(y) = b_d y^d$. This implies $P(x) = a_d(x + \frac{a_{d-1}}{da_d})^d$. Let $\frac{a_{d-1}}{da_d} = \frac{s}{r}$ with integers r, s where $r \geqslant 1$ and $(r, s) = 1$. Since P has integer coefficients, we need $r^d | a_d$. Let $a_d = r^d a$. Then $P(x) = a(rx + s)^d$. It is obvious that such a polynomial satisfies the conditions.

Comment. In the proof, the use of prime and Dirichlet's Theorem can be avoided. One can easily show that each $P(x_i)$ can be expressed in the form uv_i^d where u, v_i are integers and u cannot be divisible by the d-th power of a prime (note that u depends only on P). By fixing a large integer q and by choosing a large n, we can apply the Pigeonhole Principle and assume

$x_1 \equiv x_2 \equiv \cdots \equiv x_{d+1} \pmod q$ and $v_1 \equiv v_2 \equiv \cdots \equiv v_{d+1} \pmod q$. Then the remaining proof is similar to Case 2 of the Solution.

Alternatively, we give another modification of the proof as follows.

We take a sufficiently large n and consider the corresponding positive integers $y_1, y_2, \ldots, y_n$. For each $2 \leqslant i \leqslant n$, let $\frac{Q(y_i)}{Q(y_1)} = \frac{l_i^d}{m_i^d}$.

As in Case 1, if there are d indices i such that the integers $\frac{c|Q(y_1)|}{m_i^d}$ are bounded below by a constant depending only on P, we can establish the Claim using those y_i's and complete the proof. Similarly, as in Case 2, if there are d indices i such that the integers $|m_i y_i - l_i y_1|$ are bounded below, then the proof goes the same. So it suffices to consider the case where $\frac{c|Q(y_1)|}{m_i^d} \leqslant M$ and $|m_i y_i - l_i y_1| \leqslant N$ for all $2 \leqslant i \leqslant n'$ where M, N are fixed constants and n' is large. Since there are only finitely many choices for m_i and $m_i y_i - l_i y_1$, by the Pigeonhole Principle, we can assume without loss of generality $m_i = m$ and $m_i y_i - l_i y_1 = t$ for $2 \leqslant i \leqslant d+2$. Then

$$\frac{Q(y_i)}{Q(y_1)} = \frac{l_i^d}{m^d} = \frac{(my_i - t)^d}{m^d y_1^d}$$

so that $Q(y_1)(my-t)^d - m^d y_1^d Q(y)$ has roots $y = y_2, y_3, \ldots, y_{d+2}$. Its degree is at most d and hence it is the zero polynomial. Therefore, $Q(y) = \frac{b_d}{m^d}(my-t)^d$. Indeed, Q does not have the term y^{d-1}, which means t should be 0. This gives the corresponding $P(x)$ of the desired form.

The two modifications of the Solution work equally well when the degree d is even.