1st Olympiad of Metropolises

Mathematics · Day 1

Problem 1. Find all positive integers n such that there exist n consecutive positive integers whose sum is a perfect square.

Problem 2. Let $a_1, \ldots, a_n$ be positive integers satisfying the inequality

$$\sum_{i=1}^{n} \frac{1}{a_i} \leq \frac{1}{2}\,.$$

Every year, the government of Optimistica publishes its *Annual Report* with n economic indicators. For each $i = 1, \ldots, n$, the possible values of the i-th indicator are $1, 2, \ldots, a_i$. The Annual Report is said to be *optimistic* if at least $n-1$ indicators have higher values than in the previous report. Prove that the government can publish optimistic Annual Reports in an infinitely long sequence.

Problem 3. Let $A_1A_2 \ldots A_n$ be a cyclic convex polygon whose circumcenter is strictly in its interior. Let B_1, B_2, ..., B_n be arbitrary points on the sides A_1A_2, A_2A_3, ..., A_nA_1, respectively, other than the vertices. Prove that

$$\frac{B_1B_2}{A_1A_3} + \frac{B_2B_3}{A_2A_4} + \ldots + \frac{B_nB_1}{A_nA_2} > 1\,.$$

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Mathematics · Day 2

Problem 4. A convex quadrilateral $ABCD$ has right angles at A and C. A point E lies on the extension of the side AD beyond D so that $\angle ABE = \angle ADC$. The point K is symmetric to the point C with respect to point A. Prove that $\angle ADB = \angle AKE$.

Problem 5. Let $r(x)$ be a polynomial of odd degree with real coefficients. Prove that there exist only finitely many (or none at all) pairs of polynomials $p(x)$ and $q(x)$ with real coefficients satisfying the equation $(p(x))^3 + q(x^2) = r(x)$.

Problem 6. In a country with n cities, some pairs of cities are connected by one-way flights operated by one of two companies A and B. Two cities can be connected by more than one flight in either direction. An AB-word w is called *implementable* if there is a sequence of connected flights whose companies' names form the word w. Given that every AB-word of length 2^n is implementable, prove that every finite AB-word is implementable. (An AB-word of length k is an arbitrary sequence of k letters A or B; e. g. $AABA$ is a word of length 4.)