33rd Balkan Mathematical Olympiad
05-10 May 2016
Tirana, Albania

Shortlisted problems
and solutions

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Note of confidentiality

The shortlisted problems should be kept strictly confidential until BMO 2017.

Contributing countries

The Organizing Committee and Problem Selection Committee of BMO 2016 wish to thank the following countries for contributing with problem proposals:

Bulgaria
Cyprus
Former Yugoslav Republic of Macedonia
Greece
Moldova
Romania
Serbia
Turkey

Problem Selection Committee

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Vice Chairman:	Enkel Hysnelaj
Members:	Llukan Puka
	Artur Baxhaku
	Elton Bojaxhiu

Contents

1 Algebra

A1.

Let a, b, c be positive real numbers. Prove that

$$\sqrt{a^3b + a^3c} + \sqrt{b^3c + b^3a} + \sqrt{c^3a + c^3b} \geq \frac{4}{3}(ab + bc + ca)$$

Solution. W.L.O.G. $a \geq b \geq c$.

$$a \geq b \geq c \Rightarrow ab \geq ac \geq bc \Rightarrow ab + ac \geq ab + bc \geq ac + bc \Rightarrow \sqrt{ab + ac} \geq \sqrt{bc + ba} \geq \sqrt{ac + bc}$$

$$\sqrt{a^3b + a^3c} + \sqrt{b^3c + b^3a} + \sqrt{c^3a + c^3b} = a\sqrt{ab + ac} + b\sqrt{bc + ba} + c\sqrt{ca + cb} \geq \quad (1)$$

$$\frac{(a+b+c)}{3}\left(\sqrt{ab + ac} + \sqrt{bc + ba} + \sqrt{ca + cb}\right) = \frac{(a+b+c)}{3}\left(\sqrt{a(b+c)} + \sqrt{b(c+a)} + \sqrt{c(a+b)}\right) \geq \quad (2)$$

$$\frac{(a+b+c)}{3}\left(\frac{2a(b+c)}{a+b+c} + \frac{2b(c+a)}{b+c+a} + \frac{c(a+b)}{c+a+b}\right) = \frac{(a+b+c)}{3}\frac{4(ab+bc+ca)}{a+b+c} = \frac{4}{3}(ab + bc + ca)$$

(1) Chebyshev's inequality

(2) GM $\geq$ HM

□

A2.

For all $x, y, z > 0$ satisfying $\frac{x}{yz} + \frac{y}{zx} + \frac{z}{xy} \leq x + y + z$, prove that

$$\frac{1}{x^2+y+z} + \frac{1}{y^2+z+x} + \frac{1}{z^2+x+y} \leq 1.$$

Solution.

By Cauchy-Schwarz inequality, we have

$$(x^2+y+z)(y^2+yz^2+zx^2) \geq (xy+yz+zx)^2$$

and hence we obtain that

$$\frac{1}{x^2+y+z} + \frac{1}{y^2+z+x} + \frac{1}{z^2+x+y} \leq \frac{2(xy^2+yz^2+zx^2)+x^2+y^2+z^2}{(xy+yz+zx)^2}. \tag{1}$$

Using the condition $\frac{x}{yz} + \frac{y}{zx} + \frac{z}{xy} \leq x + y + z$, we also have

$$x^2+y^2+z^2 \leq xyz(x+y+z)$$

and hence

$$2(x^2+y^2+z^2)+x^2y^2+y^2z^2+z^2x^2 \leq (xy+yz+zx)^2. \tag{2}$$

Finally, by AM-GM

$$x^2+z^2x^2 \geq 2zx^2$$

which yields that

$$2(x^2+y^2+z^2)+x^2y^2+y^2z^2+z^2x^2 \geq 2(xy^2+yz^2+zx^2)+x^2+y^2+z^2. \tag{3}$$

Using (1), (2) and (3), we are done. □

A3.

Find all monotonic functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the condition: for every real number x and every natural number n

$$\left|\sum_{i=1}^{n} i\left(f(x+i+1) - f^2(x+i)\right)\right| < C$$

where $C > 0$ is independent of x and $f^2(x) = f(f(x))$.

Solution. From the condition of the problem we get $\left|\sum_{i=1}^{n-1} i\left(f(x+i+1) - f^2(x+i)\right)\right| < C$. Then

$$\left|n\left(f(x+n+1) - f^2(x+n)\right)\right| = \left|\sum_{i=1}^{n} i\left(f(x+i+1) - f^2(x+i)\right) - \sum_{i=1}^{n-1} i\left(f(x+i+1) - f^2(x+i)\right)\right| < 2C$$

implying $\left|f(x+n+1) - f^2(x+n)\right| < \frac{2C}{n}$ for every real number x and every natural number n.
Let $y \in \mathbb{R}$ be arbitrary. Then there exists x such that$y = x + n$. We obtain $\left|f(y+1) - f^2(y)\right| < \frac{2C}{n}$ for every real number y and every natural number n. The last inequality holds for every natural number n from where $f(y+1) = f^2(y)$ for every $y \in \mathbb{R}$. The function f is monotonic which implies that it is an injection and the latter implies $f(y) = y + 1$. □

A4.

The positive real numbers a, b, c satisfy the equality $a+b+c=1$. For every natural number n find the minimal possible value of the expression

$$E=\frac{a^{-n}+b}{1-a}+\frac{b^{-n}+c}{1-b}+\frac{c^{-n}+a}{1-c}$$

Solution. We transform the first term of the expression E in the following way:

$$\frac{a^{-n}+b}{1-a}=\frac{1+a^n b}{a^n(b+c)}=\frac{a^{n+1}+a^n b+1-a^{n+1}}{a^n(b+c)}=\frac{a^n(a+b)+(1-a)(1+a+a^2+...+a^n)}{a^n(b+c)}$$
$$\frac{a^n(a+b)}{a^n(b+c)}+\frac{(b+c)(1+a+a^2+...+a^n)}{a^n(b+c)}=\frac{a+b}{b+c}+1+\frac{1}{a}+\frac{1}{a^2}+...+\frac{1}{a^n}$$

Analogously, we obtain

$$\frac{b^{-n}+c}{1-b}=\frac{b+c}{c+a}+1+\frac{1}{b}+\frac{1}{b^2}+...+\frac{1}{b^n}$$
$$\frac{c^{-n}+a}{1-c}=\frac{c+a}{a+b}+1+\frac{1}{c}+\frac{1}{c^2}+...+\frac{1}{c^n}$$

The expression E can be written in the form

$$E=\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}+3+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+...+\left(\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}\right)$$

By virtue of the inequalities $\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\geq 3\sqrt{\frac{a+b}{b+c}\cdot\frac{b+c}{c+a}\cdot\frac{c+a}{a+b}}=3$

and $m_k=\left(\frac{a^k+b^k+c^k}{3}\right)^{\frac{1}{k}}\leq m_1=\frac{a+b+c}{3}=\frac{1}{3}$ for every $k=-1,-2,-3,...,-n$,

we have $\frac{1}{a^m}+\frac{1}{b^m}+\frac{1}{c^m}\geq 3^{m+1}$ for every $m=1,2,...,m$ and

$$E=3+3+3^2+...+3^{n+1}=2+\frac{3^{n+2}-1}{2}=\frac{3^{n+2}+3}{3} \tag{3}$$

For $a=b=c=\frac{1}{3}$ we obtain $E=\frac{3^{n+2}+3}{3}$. So, $\min E=\frac{3^{n+2}+3}{3}$. □

Remark(PSC): The original solution received from the PSC contained some typos, the correct result is $\min E=\frac{3^{n+2}+3}{2}$.

A5.

Let a, b, c and d be real numbers such that $a + b + c + d = 2$ and
$ab + bc + cd + da + ac + bd = 0$.
Find the minimum value and the maximum value of the product $abcd$.

Solution. Let's find the minimum first.

$$a^2 + b^2 + c^2 + d^2 = (a + b + c + d)^2 - 2(ab + bc + cd + da + ac + bd) = 4$$

By AM-GM, $4 = a^2 + b^2 + c^2 + d^2 \geq 4\sqrt{|abcd|} \Rightarrow 1 \geq |abcd| \Rightarrow abcd \geq -1$.
Note that if $a = b = c = 1$ and $d = -1$, then $abcd = -1$.

We'll find the maximum. We search for $abcd > 0$.
Obviously, the numbers a, b, c and d can not be all positive or all negative.
WLOG $a, b > 0$ and $c, d < 0$. Denote $-c = x, -d = y$.
We have $a, b, x, y > 0, a + b - x - y = 2$ and $a^2 + b^2 + x^2 + y^2 = 4$. We need to find $\max(abxy)$. We get: $x + y = a + b - 2$ and $x^2 + y^2 = 4 - (a^2 + b^2)$. Since $(x + y)^2 \leq 2(x^2 + y^2)$, then $2(a^2 + b^2) + (a + b - 2)^2 \leq 8$; on the other hand, $(a + b)^2 \leq 2(a^2 + b^2) \Rightarrow (a + b)^2 + (a + b - 2)^2 \leq 8$.
Let $a + b = 2s \Rightarrow 2s^2 - 2s - 1 \leq 0 \Rightarrow s \leq \frac{\sqrt{3}+1}{2} = k$.
But $ab \leq s^2 \Rightarrow ab \leq k^2$.
Now $a+b = x+y+2$ and $a^2+b^2 = 4-(x^2+y^2)$. Since $(a+b)^2 \leq 2(a^2+b^2)$, then $2(x^2+y^2)+(x+y+2)^2 \leq 8$; on the other hand, $(x + y)^2 \leq 2(x^2 + y^2) \Rightarrow$
$\Rightarrow (x + y)^2 + (x + y + 2)^2 \leq 8$. Let $x + y = 2q \Rightarrow 2q^2 + 2q - 1 \leq 0 \Rightarrow q \leq \frac{\sqrt{3}-1}{2} = \frac{1}{2k}$.
But $xy \leq q^2 \Rightarrow xy \leq \frac{1}{4k^2}$.
In conclusion, $abxy \leq k^2 \cdot \frac{1}{4k^2} = \frac{1}{4} \Rightarrow abcd \leq \frac{1}{4}$.
Note that if $a = b = k$ and $c = d = -\frac{1}{2k}$, then $abcd = \frac{1}{4}$
In conclusion, $\min(abcd) = -1$ and $\max(abcd) = \frac{1}{4}$. □

A6.

Prove that there is no function from positive real numbers to itself, $f:(0,+\infty)\to(0,+\infty)$ such that:

$$f(f(x)+y)=f(x)+3x+yf(y) \quad \text{,for every} \quad x,y\in(0,+\infty)$$

Solution. First we prove that $f(x)\geq x$ for all $x>0$.
Indeed, if there is an $a>0$ with $f(a)<a$ then from the initial for $x=a$ and $y=a-f(a)>0$ we get that $3a+(a-f(a))f(a-f(a))=0$. This is absurd since $3a+(a-f(a))f(a-f(a))>0$.
So we have that

$$f(x)\geq x \quad \text{,for all} \quad x>0 \tag{1}$$

Then using (1) we have that $f(x)+3x+yf(x)=f(f(x)+y)\geq f(x)+y$, so

$$3x+yf(y)\geq y \quad \text{, all} \quad x,y>0 \tag{2}$$

Suppose that $yf(y)<y$ for a $y>0$ and let $-yf(y)=b>0$ then for $x=\frac{b}{4}$ we get $\frac{3b}{4}-b\geq 0$ so $b\leq 0$, absurd. So we have that $yf(y)\geq y$, for all $y>0$, and so

$$f(y)\geq 1 \quad \text{,for all} \quad y>0 \tag{3}$$

Substituting y with $f(y)$ at the initial we have that

$$f(x)+3x+f(y)f(f(y))=f(f(x)+f(y)) \tag{4}$$

and changing the roles of x,y we have that:

$$f(y)+3y+f(x)f(f(x))=f(f(x)+f(x)) \tag{5}$$

So we have $f(x)f(f(x))-f(x)-3x=f(y)f(f(y))-f(y)-3y$, which means that the function $f(x)f(f(x))-f(x)-3x$ is constant. That means that there exist a constant c such that

$$f(x)f(f(x))=f(x)+3x+c \quad \text{,for all} \quad x>0 \tag{6}$$

So we can write (6) in the form $f(x)(f(f(x))-1)=3x+c$ and since $f(f(x))>1$ we have that $3x+c\geq 0$, for all $x>0$. if $c<0$ then for $x=-\frac{c}{4}>0$ we get that $c>0$ which is absurd. So $c\geq 0$.
We write (6) in the form

$$f(f(x))=1+\frac{3x}{f(x)}+\frac{c}{f(x)} \tag{7}$$

Since $c\geq 0$ then from (7) with the help of (1) and (3) we have that $f(f(x))\leq 4+c$.
But from (1) we have that $f(f(x))\geq f(x)\geq x$, for all $x\geq 0$, and so

$$4+c\geq f(f(x))\geq x \quad \text{,for all} \quad x>0$$

Taking $x=5+c$ we get that the last one cannot hold. So there is no such a function. □

A7.

Find all integers $n \geq 2$ for which there exist the real numbers $a_k, 1 \leq k \leq n$, which are satisfying the following conditions:

$$\sum_{k=1}^{n} a_k = 0, \sum_{k=1}^{n} a_k^2 = 1 \text{ and } \sqrt{n}\cdot\Big(\sum_{k=1}^{n} a_k^3\Big) = 2(b\sqrt{n}-1), \text{ where } b = \max_{1\leq k\leq n}\{a_k\}.$$

Solution. We have: $\left(a_k + \frac{1}{\sqrt{n}}\right)^2 (a_k - b) \leq 0 \Rightarrow \left(a_k^2 + \frac{2}{\sqrt{n}}\cdot a_k + \frac{1}{n}\right)(a_k - b) \leq 0 \Rightarrow$

$$a_k^3 \leq \Big(b - \frac{2}{\sqrt{n}}\Big)\cdot a_k^2 + \Big(\frac{2b}{\sqrt{n}} - \frac{1}{n}\Big)\cdot a_k + \frac{b}{n}\forall \in \{1,2,\cdot,n\}.(k)$$

Adding up the inequalities (k) we get

$$\sum_{k=1}^{n} a_k^3 \leq \left(b - \frac{2}{\sqrt{n}}\right)\cdot\left(\sum_{k=1}^{n} a_k^2\right) + \Big(\frac{2b}{\sqrt{n}} - \frac{1}{n}\Big)\cdot\left(\sum_{k=1}^{n} a_k\right) + b \leftrightarrow$$

$$\sum_{k=1}^{n} a_k^3 \leq b - \frac{2}{\sqrt{n}} + b \leftrightarrow \sqrt{n}\cdot\left(\sum_{k=1}^{n} a_k^3\right) \leq 2(b\sqrt{n}-1).$$

But according to hypothesis,

$$\sqrt{n}\cdot\left(\sum_{k=1}^{n} a_k^3\right) = 2(b\sqrt{n}-1).$$

Hence is necessarily that

$$a_k^3 = \left(b - \frac{2}{\sqrt{n}}\right)\cdot a_k^2 + \Big(\frac{2b}{\sqrt{n}} - \frac{1}{n}\Big)\cdot a_k + \frac{b}{n}\forall k \in \{1,2,\cdots,n\} \leftrightarrow$$

$$\Big(a_k + \frac{1}{\sqrt{n}}\Big)^2 (a_k - b) = 0 \forall k \in \{1,2,\cdots,n\} \leftrightarrow a_k \in \Big\{-\frac{1}{\sqrt{n}}, b\Big\}\forall k \in \{1,2,\cdots,n\}$$

We'll prove that $b > 0$. Indeed, if $b < 0$ then $0 = \sum_{k=1}^{n} a_k \leq nb < 0$, which is absurd.

If $b = 0$, since $\sum_{k=1}^{n} a_k = 0$, then $a_k = 0 \forall k \in \{1,2,\cdots,n\} \Rightarrow 1 = \sum_{k=1}^{n} a_k^2 = 0$, which is absurd.

In conclusion $b > 0$.

If $a_k = -\frac{1}{\sqrt{n}}\forall k \in \{1,2,\cdots,n\}$ then $\sum_{k=1}^{n} a_k = -\sqrt{n} < 0$, which is absurd and similarly if

$a_k = b\forall k \in \{1,2,\cdots,n\}$ then $\sum_{k=1}^{n} a_k = nb > 0$, which is absurd. Hence $\exists m \in \{1,2,\cdots,n-1\}$

such that among the numbers a_k we have $n-m$ equal to $-\frac{1}{\sqrt{n}}$ and m equal to b. We get $\begin{cases} -\frac{n-m}{\sqrt{n}} + mb = 0 \\ \frac{n-m}{n} + mb^2 = 1 \end{cases}$.

From here, $b = \frac{n-m}{m\sqrt{n}} \Rightarrow \frac{n-m}{n} + \frac{(n-m)^2}{mn} = 1 \Rightarrow$
$\Rightarrow n - m = m \Rightarrow m = \frac{n}{2}$. Hence n is even.

Conversely, for any even integer $n \geq 2$ we get that there exist the real numbers $a_k, 1 \leq k \leq n$, such that

$$\sum_{k=1}^{n} a_k = 0, \sum_{k=1}^{n} a_k^2 = 1 \text{ and } \sqrt{n}\cdot\left(\sum_{k=1}^{n} a_k^3\right) = 2(b\sqrt{n}-1), \text{ where } b = \max_{1\leq k\leq n}\{a_k\}$$

(We may choose for example $a_1 = \cdots = a_{\frac{n}{2}} = -\frac{1}{\sqrt{n}}$ and $a_{\frac{n}{2}+1} = \cdots = a_n = \frac{1}{\sqrt{n}}$). □

A8.

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ for which $f(g(n)) - g(f(n))$ is independent on n for any $g : \mathbb{Z} \to \mathbb{Z}$.

Solution. First observe that if $f(n) = n$, then $f(g(n)) - g(f(n)) = 0$. Therefore the identity function satisfies the problem condition.

If there is n_0 with $f(n_0) \neq n_0$, consider the characteristic function g that is defined as $g(f(n_0)) = 1$ and $g(n) = 0$ for $n \neq f(n_0)$. In this case, let $n \neq f(n_0)$ be arbirtrary. One has $f(g(n)) - g(f(n)) = f(g(n_0)) - g(f(n_0)) \Rightarrow f(0) - g(f(n)) = f(0) - g(f(n_0)) \Rightarrow g(f(n)) = 1 \Rightarrow f(n) = f(n_0)$.

Now, consider the very similar function g that is defined as $g(f(n_0)) = a$ and $g(n) = b$ for $n \neq f(n_0)$, where a, b are integers with $a \neq b \neq f(n_0)$. We have chosen a, b so as to ensure that $f(a) = f(n_0) = f(b)$. Now, we find that $f(g(f(n_0))) - g(f(f(n_0))) = f(g(n_0)) - g(f(n_0)) \Rightarrow f(a) - g(f(f(n_0))) = f(b) - g(f(n_0)) \Rightarrow g(f(f(n_0))) = g(f(n_0)) = a \Rightarrow f(f(n_0)) = f(n_0)$.

In summary, $f(n) = f(n_0)$ for any $n \neq f(n_0)$ and $f(f(n_0)) = f(n_0)$. Therefore f is a constant function. Let us now see that a constant function indeed satisfies the problem condition: If $f(n) = c$ for all n, $f(g(n)) - g(f(n)) = c - g(c)$ is independent of n.

The answers are the identity function $f(n) = n$ and the constant functions $f(n) = c$. $\square$

Combinatorics

C1.

Let positive integers K and d be given. Prove that there exists a positive integer n and a sequence of K positive integers $b_1, b_2, \ldots, b_K$ such that the number n is a d-digit palindrome in all number bases $b_1, b_2, \ldots, b_K$.

Solution. Let a positive integer d be given. We shall prove that, for each large enough n, the number $(n!)^{d-1}$ is a d-digit palindrome in all number bases $\frac{n!}{i} - 1$ for $1 \leqslant i \leqslant n$. In particular, we shall prove that the digit expansion of $(n!)^{d-1}$ in the base $\frac{n!}{i} - 1$ is

$$\left\langle i^{d-1}\binom{d-1}{d-1}, i^{d-1}\binom{d-1}{d-2}, i^{d-1}\binom{d-1}{d-3}, \ldots, i^{d-1}\binom{d-1}{1}, i^{d-1}\binom{d-1}{0}\right\rangle_{\frac{n!}{i}-1}.$$

We first show that, for each large enough n, all these digits are smaller than the considered base, that is, they are indeed digits in that base. It is enough to check this assertion for $i = n$, that is, to show the inequality $n^{d-1}\binom{d-1}{j} < (n-1)! - 1$. However, since for a fixed d the right-hand side clearly grows faster than the left-hand side, this is indeed true for all large enough n.

Everything that is left is to evaluate:

$$\sum_{j=0}^{d-1} i^{d-1}\binom{d-1}{j}\left(\frac{n!}{i} - 1\right)^j = i^{d-1}\sum_{j=0}^{d-1}\binom{d-1}{j}\left(\frac{n!}{i} - 1\right)^j = i^{d-1}\sum_{j=0}^{d-1}\binom{d-1}{j}\left(\frac{n!}{i} - 1\right)^j$$

$$= i^{d-1}\left(\frac{n!}{i} - 1 + 1\right)^{d-1} = (n!)^{d-1},$$

which completes the proof. $\square$

C2.

There are 2016 costumers who entered a shop on a particular day. Every customer entered the shop exactly once. (i.e. the customer entered the shop, stayed there for some time and then left the shop without returning back.)

Find the maximal k such that the following holds:

There are k customers such that either all of them were in the shop at a specific time instance or no two of them were both in the shop at any time instance.

Solution. We show that the maximal k is 45.

First we show that no larger k can be achieved: We break the day at 45 disjoint time intervals and assume that at each time interval there were exactly 45 costumers who stayed in the shop only during that time interval (except in the last interval in which there were only 36 customers). We observe that there are no 46 people with the required property.

Now we show that $k = 45$ can be achieved: Suppose that customers $C_1, C_2, ..., C_{2016}$ visited the shop in this order. (If two or more customers entered the shop at exactly the same time then we break ties arbitrarily.)

We define groups $A_1, A_2, ...$ of customers as follows: Starting with C_1 and proceeding in order, we place customer C_j into the group A_i where i is the smallest index such that A_i contains no customer $C_{j'}$ with $j' < j$ and such that $C_{j'}$ was inside the shop once C_j entered it.

Clearly no two customers who are in the same group were inside the shop at the exact same time. So we may assume that every A_i has at most 45 customers. Since $44 \cdot 45 < 2016$, by the pigeonhole principle there must be at least 45 (non-empty) groups.

Let C_j be a person in group A_{45} and suppose that C_j entered the shop at time t_j . Since we placed C_j in group A_{45} this means that for each $i < 45$, there is a $j_i < j$ such that $C_{j_i} \in A_i$ and C_{j_i} is still inside the shop at time t_j .

Thus we have found a specific time instance, namely t_j , during which at least 45 customers were all inside the shop.

Note: Instead of asking for the maximal k, an easier version is the following:

Show that there are 45 customers such that either all of them were in the shop at a specific time instance or no two of them were both in the shop at any time instance. □

C3.

The plane is divided into unit squares by means of two sets of parallel lines. The unit squares are coloured in 1201 colours so that no rectangle of perimeter 100 contains two squares of the same colour. Show that no rectangle of size 1×1201 contains two squares of the same colour.

Solution. Let the centers of the unit squares be the integer points in the plane, and denote each unit square by the coordinates of its center.

Consider the set D of all unit squares (x, y) such that $|x| + |y| \leq 24$. Any translate of D is called a *diamond*.

Since any two unit squares that belong to the same diamond also belong to some rectangle of perimeter 100, a diamond cannot contain two unit squares of the same colour. Since a diamond contains exactly $24^2 + 25^2 = 1201$ unit squares, a diamond must contain every colour exactly once.

Choose one colour, say, green, and let a_1, a_2, ... be all green unit squares. Let P_i be the diamond of center a_i. We will show that no unit square is covered by two P's and that every unit square is covered by some P_i.

Indeed, suppose first that P_i and P_j contain the same unit square b. Then their centers lie within the same rectangle of perimeter 100, a contradiction.

Let, on the other hand, b be an arbitrary unit square. The diamond of center b must contain some green unit square a_i. The diamond P_i of center a_i will then contain b.

Therefore, P_1, P_2, ... form a covering of the plane in exactly one layer. It is easy to see, though, that, up to translation and reflection, there exists a unique such covering. (Indeed, consider two neighbouring diamonds. Unless they fit neatly, uncoverable spaces of two unit squares are created near the corners: see Fig. 1.)

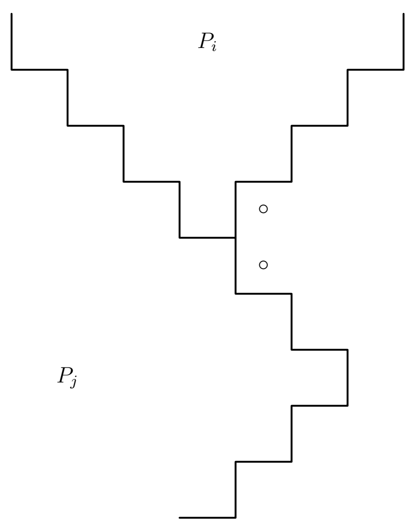


Figure 1:

Without loss of generality, then, this covering is given by the diamonds of centers (x, y) such that $24x + 25y$ is divisible by 1201. (See Fig. 2 for an analogous covering with smaller diamonds.) It follows from this that no rectangle of size 1×1201 can contain two green unit squares, and analogous reasoning works for the remaining colours.

□

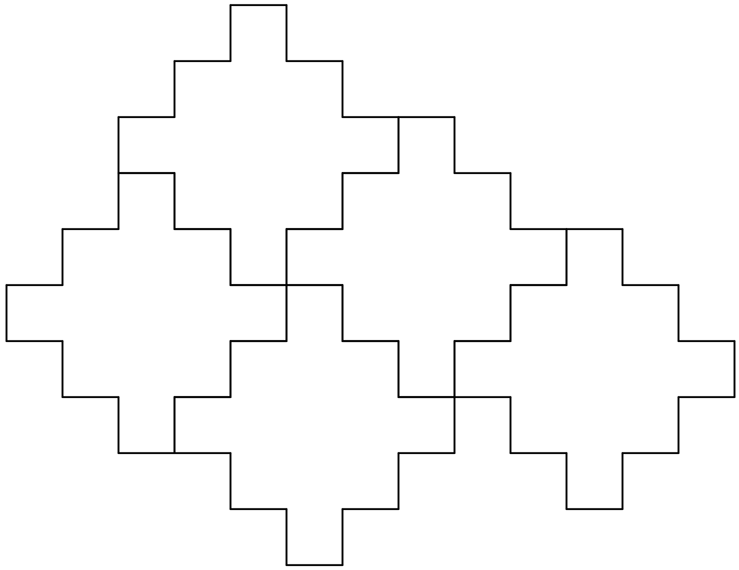


Figure 2:

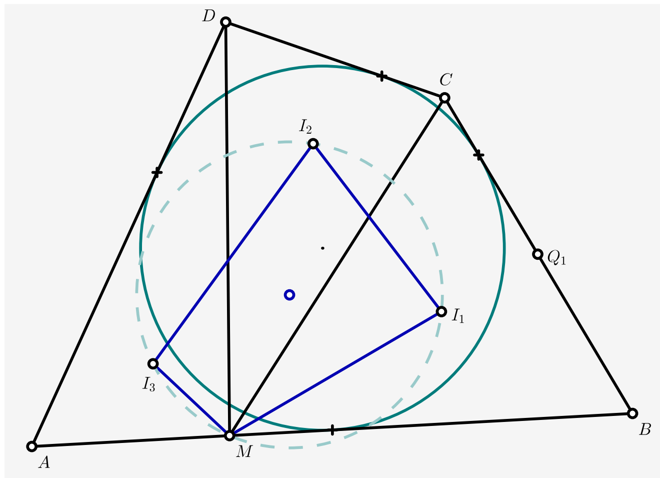
Remark(PSC): The number of the unit squares in a diamond can be evaluated alternatively with the formula

$$2 \times (1 + 3 + 5 + ... + 47) + 49 = 2 \times 24^2 + 49 = 1201$$

Geometry

G1.

The point M lies on the side AB of the circumscribed quadrilateral $ABCD$. The points I_1, I_2, and I_3 are the incenters of $\triangle MBC$, $\triangle MCD$, and $\triangle MDA$. Show that the points M, I_1, I_2, and I_3 lie on a circle.

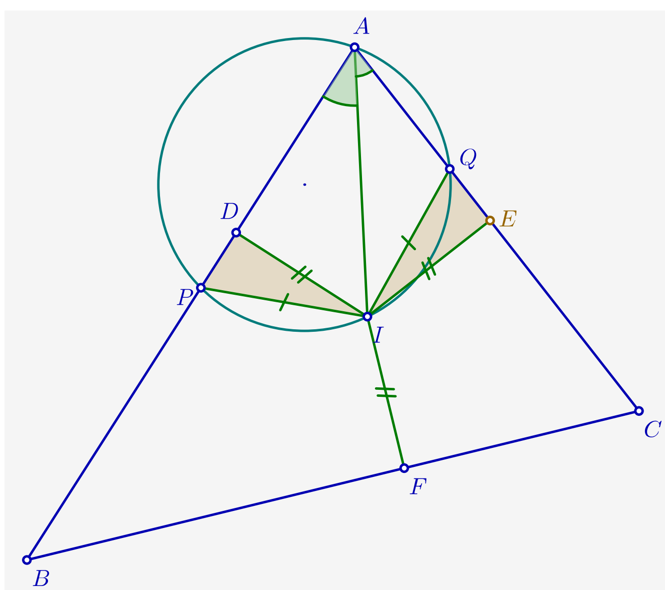


Solution. *Lemma.* Let I be the incenter of $\triangle ABC$ and let the points P and Q lie on the lines AB and AC. Then the points A, I, P, and Q lie on a circle if and only if

$$\overline{BP} + \overline{CQ} = BC$$

where $\overline{BP}$ equals $|BP|$ if P lies in the ray $BA^{\rightarrow}$ and $-|BP|$ if it does not, and similarly for $\overline{CQ}$.

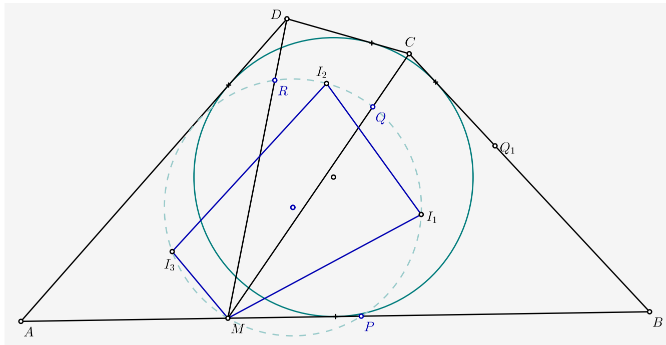
Proof of the lemma. We shall only consider the case when P and Q lie in the segments AB and AC. All other cases are treated analogously.



Suppose that A, I, P, and Q lie on a circle. Let D and E be the contact points of the incircle of $\triangle ABC$ with AB and AC. We have that $\angle PIQ = 180^\circ - \alpha$, so $\angle DIP = \angle EIQ$ and, therefore, $\triangle DIP \simeq \triangle EIQ$. This gives us $DP = EQ$ and $BP + CQ = BD + CE = BC$, as needed.

The converse is established by following the foregoing chain of inequalities in reverse. $\square$

Let the circumcircle of $\triangle MI_1I_3$ meet the lines AB, CM, and DM for the second time at P, Q, and R. By the lemma, $\overline{BP}+\overline{CQ}=BC$ and $\overline{DR}+\overline{AP}=DA$. Therefore, $\overline{CQ}+\overline{DR}=BC+DA-\overline{BP}-\overline{AP}=BC+DA-AB$. Since $ABCD$ is circumscribed, this is equal to CD, and, by the lemma, the proof is complete.



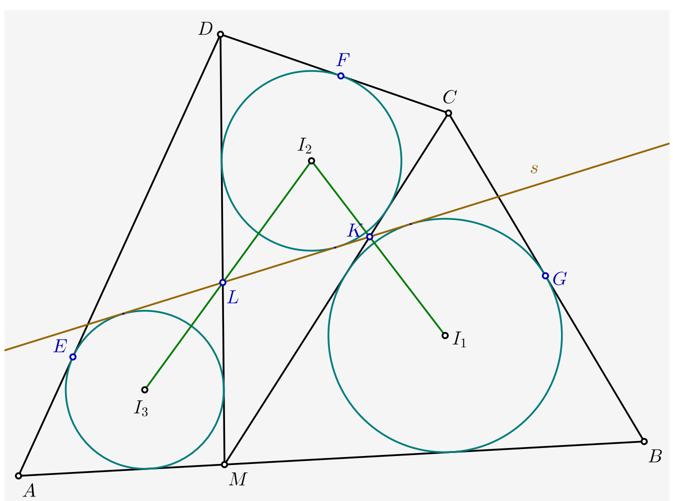
Second solution. Let ω_1, ω_2, and ω_3 be the incircles of $\triangle MBC$, $\triangle MCD$, and $\triangle MDA$.

The common internal tangent t_1 of ω_1 and ω_2 equals
[tangent from M to ω_2] − [tangent from M to ω_1] $= \frac{1}{2}(MC+MD-CD-MB-MC+BC)$.

Analogously, the common internal tangent t_2 of ω_2 and ω_3 equals
[tangent from M to ω_2] − [tangent from M to ω_3] $= \frac{1}{2}(MC+MD-CD-MD-MA+DA)$.

Finally, the common external tangent t_3 of ω_1 and ω_3 equals
[tangent from M to ω_1] − [tangent from M to ω_3] $= \frac{1}{2}(MB+MC-BC+MD+MA-DA)$.

Since $ABCD$ is circumscribed, we have $AB+CD=BC+DA$, and, therefore, $t_1+t_2=t_3$. It follows from this that ω_1, ω_2, and ω_3 have a common tangent s (which separates ω_2 from ω_1 and ω_3).



Let $\triangle MKL$ be the triangle formed by the lines MC, MD, and s. Then, since I_1I_2 and I_2I_3 are external angle bisectors in it, we have $\angle I_1I_2I_3 = 90^\circ - \frac{1}{2}\angle KML = 180^\circ - \angle I_1MI_3$ and, therefore, $MI_1I_2I_3$ is cyclic. □

Remark(PSC): In the first solution, the angle α refers to the angle $\angle BAC$.

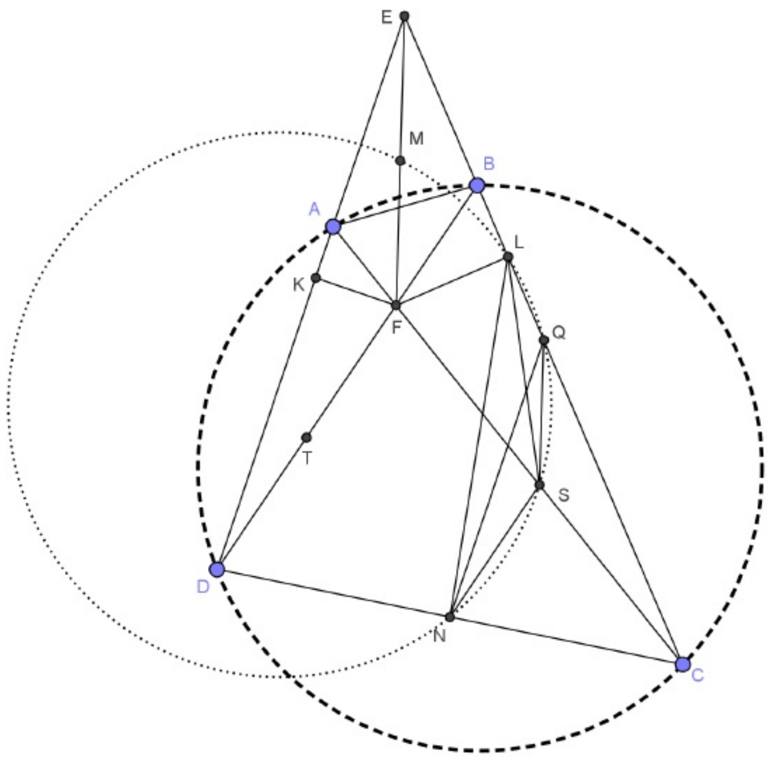
Remark(PSC): Referring to solution 1, the inequality $t_3 \le t_1+t_2$ holds, the equality is possible if and only if the circles ω_1, ω_2, and ω_3 have a common tangent.

Remark(PSC): This problem was selected as G1 relative to the other Geometry problems proposed. PSC thinks the difficulty level of this problem is medium.

G2.

Let $ABCD$ be a cyclic quadrilateral, with $AB < CD$, whose diagonals intersect at the point F and AD, BC intersect at the point E . Let also K, L be the projections of F onto the sides AD, BC respectively, and M, S, T be the midpoints of EF, CF, DF . Prove that the second intersection point of the circumcircles of the triangles MKT, MLS lies on the side CD.

Solution. Let N be the midpoint of CD. We will prove that the circumcircles of the triangles MKT, MLS pass through N.
We will prove first that the circumcircle of MLS passes through N.
Let Q be the midpoint of EC . Note that the circumcircle of MLS is the **Euler circle** of the triangle EFC, so it passes also through Q. (*)



We will prove that

$$\angle SLQ = \angle QNS \tag{1}$$

Indeed, since FLC is right-angled and LS is its median, we have that $SL = SC$ and

$$\angle SLC = \angle SCL = \angle ABC \tag{2}$$

In addition, since N, S are the midpoints of DC, FC we have that $SN // FD$.
And finally, Q, S are the midpoints of EC, CD , so $QN // ED$.
It follows that the angles $\angle EDB$ and $\angle QNS$ have parallel sides, and since $AB < CD$, they are acute, and as a result we have that

$$\angle EDB = \angle QNS \tag{3}$$

But, from the cyclic quadrilateral $ABCD$, we get that

$$\angle EDB = \angle ACB \tag{4}$$

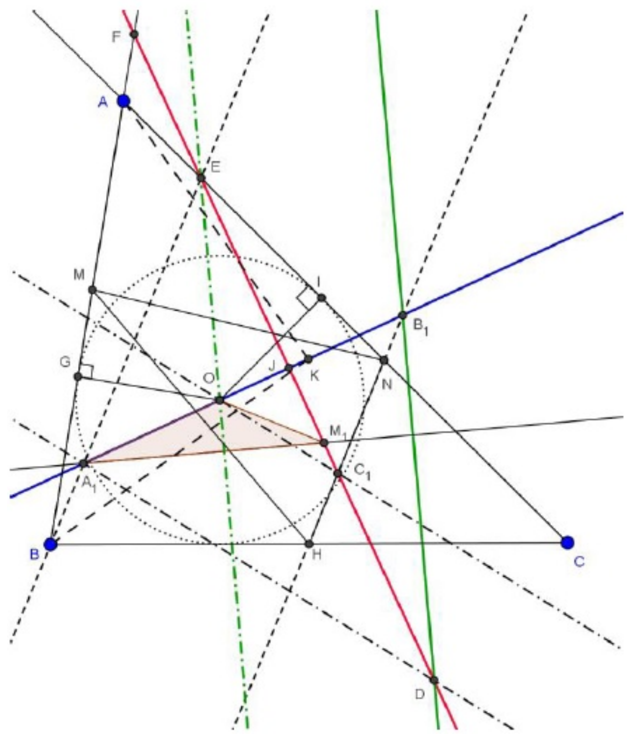
Now, from (2),(3) and (4) we obtain immediately (1), so $\angle SLQ = \angle QNS$ and the quadrilateral $LNSQ$ is cyclic. Since from (*) this circle passes also through M , we get that the points M, L, Q, S, N are co-cylic and this means that the circumcircle of MLS passes through N.

Similarly, that the circumcircle of MKT passes also through N and we have the desired. $\square$

G3.

Given that ABC is a triangle where $AB < AC$. On the half-lines BA and CA we take points F and E respectively such that $BF = CE = BC$. Let M, N and H be the mid-points of the segments BF, CE and BC respectively and K and O be the circumcircles of the triangles ABC and MNH respectively. We assume that OK cuts BE and HN at the points A_1 and B_1 respectively and that C_1 is the point of intersection of HN and FE. If the parallel line from A_1 to OC_1 cuts the line FE at D and the perpendicular from A_1 to the line DB_1 cuts FE at the point M_1, prove that E is the orthocenter of the triangle A_1OM_1.

Solution. The circumcenter of the triangle $\triangle MNH$ coincides with the incentre of the triangle $\triangle ABC$ because the triangles $\triangle BMH$ and $\triangle NHC$ are isosceles and therefore the perpendiculars of the MH, HN are also the bisectors of the angles $\angle ABC, \angle ACB$, respectively.



Let G, I be the points of tangents of the incircle (O, r) of the triangle $\triangle ABC$ with the sides AB and AC respectively. Now if a, b, c are the sides of the triangle $\triangle ABC$ and s the semiperimeter of the triangle , we have

$$OF^2 = OG^2 + FG^2 = r^2 + (a - s + b)^2$$

and

$$OE^2 = OI^2 + EI^2 = r^2 + (a - s + c)^2$$

Then

$$OF^2 - OE^2 = \alpha(b - c) \tag{1}$$

Applying two times the theorem of Stewarts at the triangles $\triangle KFB$ and $\triangle KAC$ we get

$$FA \cdot KA^2 + c \cdot KF^2 = a \cdot KA^2 + ac \cdot FA \quad \text{or} \quad KF^2 = KA^2 + a(a - c) \tag{2}$$

and

$$EA \cdot KE^2 + \alpha \cdot KA^2 = b \cdot KE^2 + ab \cdot EA \quad \text{or} \quad KE^2 = KA^2 + a(b - a) \tag{3}$$

From (2) and (3) we have

$$KF^2 - KE^2 = a(a - c) - a(b - a) = \alpha(b - c) \tag{4}$$

From (1),(4) , because $OF^2 - OE^2 = KF^2 - KE^2$ we have that $FE \perp OK$.

Let J the point of intersection of FE, OK.
Because $A_1D//OC_1$ we have

$$\frac{JO}{JA_1} = \frac{JC_1}{JD}$$

And since $A_1E//HN$ we get

$$\frac{JE}{JA_1} = \frac{JC_1}{JB_1}$$

Therefore, we have

$$\frac{JO}{JE} = \frac{JB_1}{JD}$$

Thus, from the inverse of Thales theorem we have that $EO//DB_1$, So

$$AM_1 \perp EO$$

Consequently, the point E is the orthocenter of the triangle $\triangle A_1OM_1$ □

Remark(PSC): Here in the solution the side BC has been referred as α and a, which are equivalent.

Number theory

N1.

Find all natural numbers n for which $1^{\phi(n)} + 2^{\phi(n)} + \ldots + n^{\phi(n)}$ is coprime with n.

Solution. Consider the given expression $\pmod p$ where $p \mid n$ is a prime number. $p \mid n \Rightarrow p-1 \mid \phi(n)$, thus for any k that is not divisible by p, one has $k^{\phi(n)} \equiv 1 \pmod p$. There are $n - \frac{n}{p}$ numbers among $1, 2, \ldots, n$ that are not divisible by p. Therefore

$$1^{\phi(n)} + 2^{\phi(n)} + \ldots + n^{\phi(n)} \equiv -\frac{n}{p} \pmod p$$

If the given expression is coprime with n, it is not divisible by p, so $p \nmid \frac{n}{p} \Rightarrow p^2 \nmid n$. This is valid for all prime divisors p of n, thus n must be square-free. On the other hand, if n is square-free, one has $p^2 \nmid n \Rightarrow p \nmid \frac{n}{p}$, hence the given expression is not divisible by p. Since this is valid for all prime divisors p of n, the given two numbers are indeed coprime.

The answer is square-free integers. $\square$

N2.

Find all odd natural numbers n such that $d(n)$ is the largest divisor of the number n different from n ($d(n)$ is the number of divisors of the number n including 1 and n).

Solution. From $d(n)|n$, $\frac{n}{d(n)}|n$ one obtains $\frac{n}{d(n)} \leq d(n)$.
Let $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_s^{\alpha_s}$ where $p_i, 1 \leq i \leq s$ are prime numbers. The number n is odd from where we get $p_i > 2, 1 \leq i \leq s$. The multiplicativity of the function $d(n)$ implies $d(n) = (1+\alpha_1)...(1+\alpha_s)$.
From Bernoulli inequality, for every $p_i > 3$ from the factorization of n, one obtain

$$p_i^{\frac{\alpha_i}{2}} = (1 + (p_i - 1))^{\frac{\alpha_i}{2}} \geq 1 + \frac{\alpha_i}{2}(p_i - 1) > 1 + \alpha_i \quad \text{and} \quad 3^{\frac{\beta}{2}} \geq 1 + \beta$$

The equality holds when $\beta = 0, \beta = 2$, and strict inequality for $\beta > 2$. If $\beta = 1$ and there is no prime number in the factorization of n, then $n = 3, d(n) = 2$, which is not a solution of the problem. If $\beta = 1$ and there exist another prime in the factorization of n, then $n = 3p_2^{\alpha_2}...p_s^{\alpha_s} > 4(1+\alpha_2)^2...(1+\alpha_s)^2$. Indeed, if there is $p_i \geq 7$ in the factorization, then $3p_i^{\alpha_i} > 4(1+\alpha_i)^2$ for every natural number α_i.
If the power of 5 in the factorization is bigger than 1, then $3 \cdot p_i^{\alpha_i} > 4(1+\alpha_i)^2$. Hence, if the power of 5 is 1, then $n = 3 \cdot 5 = 15, d(n) = 4$, which again is not a solution. The conclusion is that $\beta \neq 1$.
Finaly, we obtain $\sqrt{n} \geq d(n)$ and equality holds when there is no $p_i > 3$ in the factorization of n and $n = 3^2 = 9$. $\square$

N3.

Find all the integer solutions (x, y, z) of the equation

$$(x+y+z)^5 = 80xyz(x^2+y^2+z^2).$$

Solution. We directly check the identity

$$(x+y+z)^5 - (-x+y+z)^5 - (x-y+z)^5 - (x+y-z)^5 = 80xyz(x^2+y^2+z^2).$$

Therefore, if integers x, y and z satisfy the equation from the statement, we then have

$$(-x+y+z)^5 + (x-y+z)^5 + (x+y-z)^5 = 0.$$

By Fermat's theorem at least one of the parenthesis equals 0. Let, w.l.o.g., $x = y + z$. Then the previous equation reduces to $(2z)^5 + (2y)^5 = 0$, which is equivalent to $y = -z$. Therefore, the solution set of the proposed equation is

$$(x, y, z) \in \{(0, t, -t), (t, 0, -t), (t, -t, 0) : t \in \mathbb{Z}\}.$$

□

Remark(PSC): Due to the volume of calculations that are involved in the solution of this problem, a more simplified version can be:

Find all the integer solutions (x, y, z) of the equation

$$(x+y+z)^3 = 24xyz.$$

The solution would be similar to the one proposed in the original problem using the identity

$$(x+y+z)^3 - (-x+y+z)^3 - (x-y+z)^3 - (x+y-z)^3 = 24xyz.$$

N4.

Find all monic polynomials f with integer coefficients satisfying the following condition:
There exists a positive integer N such that for every prime $p > N, p$ divides $2\left(f(p)\right)! + 1$.

Solution. From the divisibility relation $p | 2\left(f(p)\right)! + 1$ we conclude that:

$$f(p) < p, \quad \text{for all primes} \quad p > N \qquad (*)$$

In fact, if for some prime number p we have $f(p) \geq p$, then $p | \left(f(p)\right)!$ and then $p | 1$, which is absurd.
Now suppose that $\deg f = m > 1$. Then $f(x) = x^m + Q(x), \deg Q(x) \leq m - 1$ and so $f(p) = p^m + Q(p)$. Hence for some large enough prime number p holds that $f(p) > p$, which contradicts (*). Therefore we must have $\deg Q(x) = 1$ and $Q(x) = x - a$, for some positive integer a. Thus the given condition becomes:

$$p | 2(p - a)! + 1 \qquad (1)$$

But we have (using Wilsons theorem)

$$2(p-3)! \equiv -(p-3)!(p-2) \equiv -(p-2)! \equiv -1 \pmod{p}$$
$$\Rightarrow p | 2(p-3)! + 1 \qquad (2)$$

From (1) and (2) we get $(p - 3)! \equiv (p - a)! \pmod{p}$. Since $p - 3 < p$ and $p - a < p$, we conclude that $a = 3$ and $f(p) = p - 3$, for every prime $p > N$. Finally, since the number of all these primes is infinite we conclude that $f(x) = x - 3$, for all x. □

Remark(PSC): There is a typo in the original solution received, the part of the solution $\deg Q(x) = 1$ and $Q(x) = x - a$ should be replaced with $\deg f(x) = 1$ and $f(x) = x - a$.

N5.

A positive integer n is *downhill* if its decimal representation $\overline{a_k a_{k-1} \dots a_0}$ satisfies $a_k \geq a_{k-1} \geq \dots \geq a_0$. A real-coefficient polynomial P is *integer-valued* if $P(n)$ is an integer for all integer n, and *downhill-integer-valued* if $P(n)$ is an integer for all downhill positive integers n. Is it true that every downhill-integer-valued polynomial is also integer-valued?

Solution. No, it is not.

A downhill number can always be written as $a - b_1 - b_2 - \dots - b_9$, where a is of the form $\overline{99\dots99}$ and each b_i either equals 0 or is of the form $\overline{11\dots11}$.

Let n be a positive integer. The numbers of the form $\overline{99\dots99}$ yield at most n different remainders upon division by 2^n, as do the numbers of the form $\overline{11\dots11}$. Therefore, downhill numbers yield at most $n(n+1)^9$ different remainders upon division by 2^n.

Let n be so large that $n(n+1)^9 < 2^n$. ($n = 63$ works: $63 \times 64^9 < 64^{10} = 2^{60} < 2^{63}$.) Let $0 \leq r < 2^n$ be such that no downhill number is congruent to r modulo 2^n.

Consider the polynomial

$$P(x) = \frac{1}{2 \times (2^n - 1)!} \prod_{1 \leq i < 2^n} (x - r + i).$$

We have that $P(r) = \frac{1}{2}$ is not an integer.

Let, then, x be a downhill number. The number $(x-r+1)\dots(x-r+2^n-1)$ is a multiple of $(2^n-1)!$ (as a product of $2^n - 1$ consecutive integers); therefore, $2P(x)$ is an integer. On the other hand, the number $(x-r)(x-r+1)\dots(x-r+2^n-1)$ ia a multiple of $2^n!$ (as a product of 2^n consecutive integers); therefore, $2(x-r)P(x)$ is an integer multiple of 2^n. Since x is downhill, $x - r$ is not divisible by 2^n. Therefore, $2P(x)$ is even and $P(x)$ is an integer.

Alternative version. A positive integer n is *uphill* if its decimal representation $\overline{a_k a_{k-1} \dots a_0}$ satisfies $a_k \leq a_{k-1} \leq \dots \leq a_0$. A real-coefficient polynomial P is *integer-valued* if $P(n)$ is an integer for all integer n, and *uphill-integer-valued* if $P(n)$ is an integer for all uphill positive integers n. Is it true that every uphill-integer-valued polynomial is also integer-valued?

Solution. First we show that no *uphill* number is congruent to 10 modulo 11.

To this end, notice that an uphill number can always be written as $b_1 + b_2 + \dots + b_m$, where $m \leq 9$, $b_1 \leq b_2 \leq \dots \leq b_m$, and each b_i is of the form $\overline{11\dots11}$. Since the remainder of each b_i modulo 11 is either 0 or 1, the remainder of $b_1 + b_2 + \dots + b_m$ modulo 11 is at most 9, as required.

Consider, then, the polynomial

$$P(x) = \frac{1}{11} x(x-1)\cdots(x-9)$$

It value is an integer at every uphill number; however, $P(10)$ is not an integer. □

Remark(PSC): PSC believes the alternative version of this problem has medium as level of difficulty.