Serbian Mathematical Olympiad 2015

for high school students

Belgrade, March 27–28

Problems and Solutions

Edited by Dušan Djukić

Cover photo: A river house on Drina, near Bajina Bašta

SHORT HISTORY AND SYSTEM

Mathematical competitions in Serbia have been held since 1958. In the first years only republic competitions within the former Yugoslavia, which Serbia was a part of, were held. The first Federal Mathematical Competition in Yugoslavia was held in Belgrade in 1960, and since then it was held regularly every year, skipping only 1999 for a non-mathematical reason. The system has undergone relatively few changes. The earliest Federal Competitions were organized for 3rd and 4th grades of high school only; 2nd grade was added in 1970, and 1st grade in 1974. Since 1982, 3rd and 4th grades compose a single category. After the breakdown of the old Yugoslavia in 1991, the entire system was continued in the newly formed FR Yugoslavia, later renamed Serbia and Montenegro. The separation of Montenegro finally made the federal competition senseless as such. Thus, starting with 2007, the federal competition and team selection exam are replaced by a two-day Serbian Mathematical Olympiad.

Today a mathematical competition season in Serbia consists of four rounds of increasing difficulty:

- Municipal round, held in early February. The contest consists of 5 problems, each 20 points worth, to be solved in 3 hours. Students who perform well qualify for the next round (50-60 points are usually enough).

- Regional round, held in late February in the same format as the municipal round. Each student's score is added to that from the municipal round. Although the number of students qualified for the state round is bounded by regional quotas, a total score of 110-120 should suffice.

- State (republic) round, held in late March in a selected town in the country. There are roughly 200 to 300 participants. The contest consists of 5 problems in 4 hours.

- Serbian Mathematical Olympiad (SMO), held in early April in a selected place in the country. The participants are selected through the state round: 28 from A category (distribution among grades: 4+8+8+8), 3 from B category (0+0+1+2), plus those members of the last year's olympic team who did not manage to qualify otherwise. Six most successful contestants are invited to the olympic team.

Since 1998, contests for each grade on the three preliminary rounds are divided into categories A (specialized schools and classes) and B (others). A student from category B is normally allowed to work the problems for category A instead. On the SMO, all participants work on the same problems.

The 9-th Serbian Mathematical Olympiad for high school students took place in Belgrade on March 27–28, 2015. There were 38 students from Serbia and 4 guest students from Republika Srpska (Bosnia and Herzegovina). The average score on the contest was 15.26 points. Problems 1, 2 and 4 turned out to be rather easy, while no student solved problem 3.

The team for the 32-nd Balkan MO and 56-th IMO was selected based on the contest:

Marijana Vujadinović	Math High School, Belgrade	35 points
Ognjen Tošić	Math High School, Belgrade	30 points
Ivan Damnjanović	HS "Bora Stanković", Niš	29 points
Aleksa Milojević	Math High School, Belgrade	28 points
Aleksa Konstantinov	Math High School, Belgrade	26 points
Andjela Šarković	HS "Svetozar Marković", Niš	25 points

In this booklet we present the problems and full solutions of the Serbian Mathematical Olympiad and the Balkan Mathematical Olympiad.

Serbian MO 2015 – Problem Selection Committee

- Vladimir Baltić
- Bojan Bašić *(chairman)*
- Dušan Djukić
- Miljan Knežević
- Miloš Milosavljević
- Nikola Petrović
- Marko Radovanović
- Miloš Stojaković

SERBIAN MATHEMATICAL OLYMPIAD

for high school students

Belgrade , 27.03.2015.

First Day

1. Let $ABCD$ be an inscribed quadrilateral and let M, N, P and Q be the midpoints of the sides DA, AB, BC and CD, respectively. The diagonals AC and BD intersect at point E, and the circumcircles of $\triangle EMN$ and $\triangle EPQ$ meet at point $F \neq E$. Prove that $EF \perp AC$. *(Dušan Djukić)*

2. A natural number k is given. For $n \in \mathbb{N}$ we define $f_k(n)$ as the smallest integer greater than kn such that $nf_k(n)$ is a perfect square. Prove that $f_k(m) = f_k(n)$ implies $m = n$. *(Nikola Petrović)*

3. A guard proposes the following game to the prisoners. All prisoners are to be taken to the prison yard, where each of them will be put a hat in one of 5 possible colors onto his head, and aligned so that each of them can see all hats but his own. The guard will then ask the first prisoner to say aloud whether he knows the color of his hat. If he answers "no", he will be publicly executed. Otherwise, he will be asked to say the color of his hat in such a way that others do not hear his answer. If the answer is correct, he will be freed, otherwise he will be publicly executed. The guard will then go on to the next prisoner in line and repeat the procedure, and so on. The prisoners may devise a strategy before the game starts, but no communication between them during the game is allowed. If there are 2015 prisoners, what is the maximal number of them that can have guaranteed freedom using an optimal strategy? *(Bojan Bašić)*

Time allowed: 270 minutes.
Each problem is worth 7 points.

SERBIAN MATHEMATICAL OLYMPIAD

for high school students

Belgrade, 28.03.2015.

Second Day

4. For a nonzero integer a, denote by $v_2(a)$ the largest nonnegative integer k such that $2^k \mid a$. Given $n \in \mathbb{N}$, determine the largest possible cardinality of a subset A of set $\{1, 2, 3, \ldots, 2^n\}$ with the following property:

for all $x, y \in A$ with $x \neq y$, $v_2(x - y)$ is even.

(Dušan Djukić)

5. Prove that the inequality

$$\frac{x-y}{xy+2y+1} + \frac{y-z}{yz+2z+1} + \frac{z-x}{zx+2x+1} \geqslant 0$$

holds for any nonnegative real numbers x, y and z. *(Dušan Djukić)*

6. Find all nonnegative integer solutions of the equation

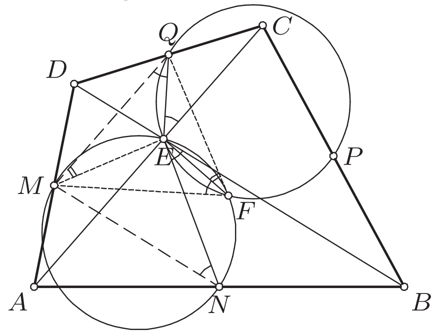
$$(2^{2015} + 1)^x + 2^{2015} = 2^y + 1.$$

(Bojan Bašić)

Time allowed: 270 minutes.
Each problem is worth 7 points.

SOLUTIONS

1. Since triangles EAB and EDC are similar, so are triangles EBN and ECQ. Thus we have $\sphericalangle MFE = \sphericalangle MNE = \sphericalangle BEN = \sphericalangle QEC = \sphericalangle EQM$ in oriented angles modulo 180°. Analogously, $\sphericalangle QFE = \sphericalangle EMQ$, which implies that F is the orthocenter of triangle EMQ. Hence $EF \perp QM \parallel AC$.



Second solution. Consider the translation $\mathcal{T}$ by $\frac{1}{2}\overrightarrow{AC}$. Then $\mathcal{T}(M) = Q$ and $\mathcal{T}(N) = P$. Denote $\mathcal{T}(E) = E'$. The similarity of triangles AED and BEC gives us $\triangle AEM \sim \triangle BEP$ and hence $\sphericalangle QE'E = \sphericalangle EMQ = \sphericalangle MEA = \sphericalangle BEP = \sphericalangle QPE$, so the point E' lies on the circle PEQ. Therefore $\mathcal{T}$ maps the circle MEN to the circle PEQ, so the line through their centers is parallel to AC, implying the problem statement.

2. Suppose that $f_k(m) = f_k(n) = q$. Let us write q in the form $q = au^2$ with $a, u \in \mathbb{N}$ and a squarefree. Since $mq = amu^2$ is a perfect square, so is am, i.e. $m = av^2$ for some $v \in \mathbb{N}$. Similarly, we have $n = aw^2$ for some $w \in \mathbb{N}$.

Since $f_k(av^2) = au^2$, u is the smallest natural number greater than $v\sqrt{k}$. Similarly, u is the smallest natural number greater than $w\sqrt{k}$, so we must have $|v\sqrt{k} - w\sqrt{k}| < 1$. However, this implies $|v - w| < \frac{1}{\sqrt{k}} < 1$ and therefore $v = w$, i.e. $m = n$.

3. Denote the colors by numbers $0, 1, 2, 3, 4$. Let B be the color of the second prisoner's hat and let S be the sum modulo 5 of hat colors of all prisoners but the first two. We shall describe a strategy of the first two prisoners which lets all the others know S, which will in turn determine their own hat color. Thus (at least) 2013 prisoners will be released.

The first prisoner answers *No* if $S \in \{0, B, B+1\}$ or $(B, S) = (4, 1)$. The second prisoner then answers *No* if $S = 0$ and answers *Yes* followed by the guess S otherwise. In this way, if both answers are *No*, the others can deduce $S = 0$. Also, if the second prisoner is released, they deduce $S = B$, whereas if he answers *Yes* and dies, they deduce $S = B + 1$ (if $B \neq 4$) or $S = 1$ (if $B = 4$).

Suppose the first prisoner answers *Yes* instead (his fate does not matter). The second then answers *Yes* if $S \in \{2, 4\}$ and guesses $S - 2$. Then $B = 0$ would imply $S \notin \{0, 1\}$, so the others would know that $S = 2$ if the second prisoner is set free, $S = 3$ if he says *No*, and $S = 4$ if he says *Yes* and misses. Finally, if $B \neq 0$, since the first answered in the affirmative, we have $S \equiv B + 2$ or $S \equiv B + 3 \pmod 4$, and the answer of the second determines the parity of S and S itself.

It remains to show that no strategy can guarantee freedom for 2014 prisoners. Suppose that such a strategy exists and consider five configurations differing in the second prisoner's hat color only. The first can say *No* in at most one case (otherwise the second will not be able to determine his hat color), and in at least three of the remaining four cases the first would have to miss, leaving the second with insufficient data again. The proof is complete.

4. We shall prove by induction on k that set A can contain at most 2^k different elements modulo 2^{2k}. This is trivial for $k = 0$; let $k > 0$. By the induction hypothesis, the elements of A give at most 2^{k-1} different residues modulo 2^{2k-2}. Suppose that they give more than 2^k residues modulo 2^{2k}. Then by the pigeonhole principle, some three elements are equal modulo 2^{2k-2}, and among these three, two differ by $2^{2k-1} \pmod{2^{2k}}$, contradiction the problem condition.

It follows that $|A| \leqslant 2^{[\frac{n+1}{2}]}$. An example with exactly $2^{[\frac{n+1}{2}]}$ elements can be obtained by including the numbers of the form $\sum_{i \in B} 4^i$, where B runs through all subsets of $\{0, 1, \dots, [\frac{n-1}{2}]\}$.

Second solution. We call a set X *happy* if $v_2(x - y)$ is even for all distinct $x, y \in X$, and *unhappy* if $v_2(x - y)$ is odd for all distinct $x, y \in X$. Let a_n and b_n be the maximal cardinalities of happy and unhappy subsets of the set $T_n = \{1, 2, \dots, 2^n\}$, respectively.

Consider a happy subset $A \subset T_n$, $n \geqslant 1$. Since $v_2(2x - 2y) = v_2(x - y) + 1$, sets $A_0 = \{\frac{x}{2} \mid x \in A,\ 2 \mid x\}$ and $A_1 = \{\frac{x-1}{2} \mid x \in A,\ 2 \nmid x\}$ are unhappy subsets of T_{n-1} having at most b_{n-1} elements each. On the other hand, if $A_0 \subset T_{n-1}$ is unhappy, then $\{2x, 2x + 1 \mid x \in A_0\} \subset T_n$ is a happy set with $2|A_0|$ elements. Therefore $a_n = 2b_{n-1}$.

Similarly, if $B \subset T_n$ is unhappy, all its elements have the same parity, so $B' = \{\lceil \frac{x}{2} \rceil \mid x \in B\} \subset T_{n-1}$ is a happy set. Conversely, if $B' \subset T_{n-1}$ happy, then $B = \{2x - 1 \mid x \in B'\} \subset T_n$ is unhappy. Therefore $b_n = a_{n-1}$.

The above relations yield $a_n = 2a_{n-2}$ for $n \geqslant 2$, so the equalities $a_0 = 1$ and $a_1 = 2$ imply $a_n = 2^{[\frac{n+1}{2}]}$.

5. Let us denote $a = \frac{x-y}{xy+2y+1}$, $b = \frac{y-z}{yz+2z+1}$ and $c = \frac{z-x}{zx+2x+1}$. Then $1 + \frac{1}{a} = \frac{xy+x+y+1}{x-y}$ and hence $\frac{a}{a+1} = \frac{x-y}{xy+x+y+1} = \frac{1}{y+1} - \frac{1}{x+1}$. Analogously, we have $\frac{b}{b+1} = \frac{1}{z+1} - \frac{1}{y+1}$ and $\frac{c}{c+1} = \frac{1}{x+1} - \frac{1}{z+1}$.

It follows from $0 < \frac{1}{x+1}, \frac{1}{y+1}, \frac{1}{z+1} < 1$ that $\frac{a}{a+1}, \frac{b}{b+1}, \frac{c}{c+1} < 1$, which means that $a+1, b+1, c+1$ are positive. Since $\frac{a}{a+1} + \frac{b}{b+1} + \frac{c}{c+1} = 0$, we have $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 3$, so the Cauchy-Schwartz inequality implies $(a+1)+(b+1)+(c+1) \geqslant 3$, i.e. $a+b+c \geqslant 0$.

Second solution. The inequality is equivalent to

$$\begin{aligned} &2(x-1)^2(y-z)^2 + 2(y-1)^2(z-x)^2 + 2(z-1)^2(x-y)^2 \\ &+9(xy^2+yz^2+zx^2-3xyz) + 3(x^2y+y^2z+z^2x-3xyz) \quad \geqslant 0, \end{aligned}$$

where all the summands are nonnegative by the mean inequality.

Remark. If the condition $x, y, z \geqslant 0$ is replaced by $x, y, z \geqslant -\varepsilon$ for any $\varepsilon > 0$, the statement is no longer valid. For example, it is false for $(x, y, z) = (-\varepsilon, 1, \frac{2}{\varepsilon})$.

6. The only solutions with $x \leqslant 1$ are $(0, 2015)$ and $(1, 2016)$.

Assume that $x > 1$. Since $2^{2015}+1$ is divisible by 3, we have $(2^{2015}+1)^x + 2^{2015} \equiv 2^{2015} \equiv 5 \pmod 9$, so $2^y \equiv 4 \pmod 9$, which gives us $y = 6k+2$ for some $k \in \mathbb{N}$. Now modulo 13 we have $2^y + 1 = (2^6)^k \cdot 2^2 \equiv \pm 4$ and $2^{2015} \equiv 7 \pmod{13}$, so $8^x + 7 \equiv \pm 4 \pmod{13}$. This is impossible, as 8^x always gives one od the remainders $1, 5, 8, 12$ modulo 13.

Second solution. For $x > 1$ we have $2^y = (2^{2015}+1)^x + 2^{2015} - 1 = (x+1)2^{2015} + \sum_{i=2}^{x} \binom{x}{i} 2^{2015i} \equiv (x+1)2^{2015} \pmod{2^{2019}}$, but $y > 2019$, so we infer $16 \mid x+1$. Now reducing the equation modulo 17 and using $2^{2015} \equiv 9 \pmod{17}$ yields $2^y \equiv 10^x + 8 \equiv 10^{15} + 8 \equiv 3 \pmod{17}$, which is impossible.

The 32-nd Balkan Mathematical Olympiad was held from May 3 to May 8 in Athens in Greece. The results of the Serbian contestants are shown below:

	1	2	3	4	Total	
Marijana Vujadinović	10	10	10	0	30	Silver medal
Ognjen Tošić	5	10	0	0	15	Bronze medal
Ivan Damnjanović	4	0	0	0	4	
Aleksa Milojević	0	10	0	0	10	H. mention
Aleksa Konstantinov	0	10	0	0	10	H. mention
Andjela Šarković	0	10	0	0	10	H. mention

To put it mildly, this is obviously not as good as we hoped. A part of the reason could be the simple inequality in problem 1 - seemingly easy for everyone except for the Serbian team. Problems 3 and 4 looked standard but nevertheless turned out to be harder. At the end, 7 contestants (6 official + 1 guest) with 31-40 points were awarded gold medals, 15 (13+2) with 24-30 points were awarded silver medals, and 55 (22+33) with 12-23 points were awarded bronze medals.

Here is the (unofficial) team ranking:

Member Countries		Guest Teams	
1. Turkey	177	Turkmenistan	127
2. Romania	171	Kazakhstan	118
3. Bulgaria	156	Greece B	97
4. Greece	133	Tajikistan	97
5. Moldova	106	Italy	90
6. Bosnia and Herzegovina	105	Saudi Arabia	79
7. Macedonia (FYR)	79	UK and Ireland	68
7. Serbia	79	Azerbaijan	61
9. Cyprus	45	Kyrgyzstan	14
10. Albania	38	Iran (independent)	8
11. Montenegro	3		

BALKAN MATHEMATICAL OLYMPIAD

Athens, Greece , 05.05.2015.

1. Let a, b and c be positive real numbers. Prove that

$$a^3b^6 + b^3c^6 + c^3a^6 + 3a^3b^3c^3 \geqslant abc(a^3b^3 + b^3c^3 + c^3a^3) + a^2b^2c^2(a^3 + b^3 + c^3).$$

(Montenegro)

2. Let ABC be a scalene triangle with incenter I and circumcircle ω. The lines AI, BI, CI intersect ω for the second time at the points D, E, F, respectively. The lines through I parallel to the sides BC, CA, AB intersect the lines EF, FD, DE at the points K, L, M, respectively. Prove that the points K, L, M are collinear. *(Cyprus)*

3. A jury of 3366 film critics are judging the Oscars. Each critic makes a single vote for his favorite actor, and a single vote for his favorite actress. It turns out that for every integer $n \in \{1, 2, \ldots, 100\}$ there is an actor or actress who has been voted for exactly n times. Show that there are two critics who voted for the same actor and for the same actress. *(Cyprus)*

4. Prove that among any 20 consecutive positive integers there exists an integer d such that for each positive integer n we have the inequality

$$n\sqrt{d} \cdot \left\{n\sqrt{d}\right\} > \frac{5}{2}$$

where $\{x\}$ denotes the fractional part of the real number x. The fractional part of a real number x is x minus the greatest integer less than or equal to x. *(Serbia)*

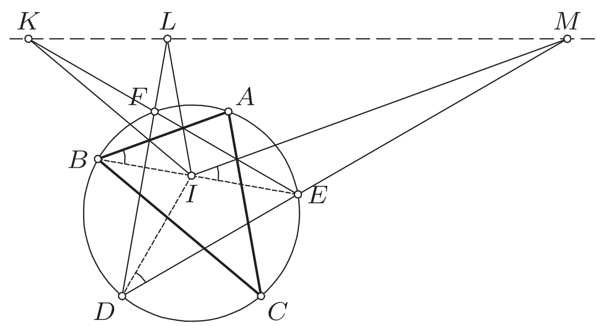
Time allowed: 270 minutes.
Each problem is worth 10 points.

SOLUTIONS

1. Setting $x = ab^2$, $y = bc^2$ and $z = ca^2$ the given inequality immediately reduces to Schur's inequality :

$$x^3 + y^3 + z^3 + 3xyz \geqslant xy^2 + yz^2 + zx^2 + x^2y + y^2z + z^2x.$$

2. Assuming the arrangement $D - E - M$, we have $\sphericalangle EIM = \sphericalangle EBA = \sphericalangle EDI$, so the line IM is tangent to the circle DEI and $MI^2 = MD \cdot ME$. Thus M lies on the radical axis s of the circle ω and the degenerate circle I. Similarly, points K and L also lie on s.



Second solution. Points $K' = BC \cap EF$, $L' = CA \cap FD$ and $M' = AB \cap DE$ are collinear by Desargues' theorem, so we have $\frac{EK'}{K'F} \cdot \frac{FL'}{L'D} \cdot \frac{DM'}{M'E} = -1$ in oriented segments. Since $\frac{EK'}{K'F} = \frac{EK}{KF} \cdot \frac{EK'}{EK} \cdot \frac{KF}{K'F} = \frac{EK}{KF} \cdot \frac{BE}{IE} \cdot \frac{IF}{CF}$, substituting this equality and the corresponding ones for L' and M' in the previous relation gives us $\frac{EK}{KF} \cdot \frac{FL}{LD} \cdot \frac{DM}{ME} = -1$. Thus the problem statement follows by Menelaus' theorem.

Remark. Both solutions still work if I is an arbitrary point in the plane of $\triangle ABC$.

3. Assume the contrary. For each $i = 1, \ldots, 100$ choose a candidate A_i who was voted for exactly i times.

The number of judges who gave both their votes for candidates in the set $\mathcal{A} = \{A_{34}, A_{35}, \ldots, A_{100}\}$ does not exceed the number of pairs actor-actress in $\mathcal{A}$, and the number of such pairs is at most $33 \cdot 34 = 1122$.

On the other hand, of the $2 \cdot 3366 = 6732$ votes, exactly $34 + 35 + \cdots + 100 = 4489$ were given to the candidates in $\mathcal{A}$. Therefore at most $6732 - 4489 = 2243$ judges could have given a vote to a candidate not in $\mathcal{A}$.

Thus, there were at most $1122 + 2243 = 3365$ judges, a contradiction.

4. Denoting $m = \left[n\sqrt{d}\right]$ we have

$$n\sqrt{d}\left\{n\sqrt{d}\right\} = n\sqrt{d}\left(n\sqrt{d} - m\right) = n\sqrt{d} \cdot \frac{dn^2 - m^2}{n\sqrt{d} + m} > n\sqrt{d} \cdot \frac{dn^2 - m^2}{2n\sqrt{d}} = \frac{dn^2 - m^2}{2}.$$

Thus it suffices to choose d so that $dn^2 - m^2 \notin \{1, 2, 3, 4\}$ holds for all $m, n \in \mathbb{N}$. This can be done by taking $d = 20k + 15 = 5(4k + 3)$ for $k \in \mathbb{N}_0$. Indeed, numbers

$m^2 + 2$ and $m^2 + 3$ are not divisible by 5, whereas $m^2 + 1$ and $m^2 + 4$ have no divisors of the form $4k + 3$; hence, none of the numbers $m^2 + 1$, $m^2 + 2$, $m^2 + 3$, $m^2 + 4$ is a multiple of d.

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