Ministry of Education, Science and Technological Development
Serbian Mathematical Society

8^{th} SERBIAN MATHEMATICAL OLYMPIAD FOR HIGH SCHOOL STUDENTS

Novi Sad – April 5, 2014

First Day

1. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$

$$f\big(xf(y) - yf(x)\big) = f(xy) - xy.$$

(Dušan Djukić)

2. Let D and E be points on sides BC and AC of a triangle ABC, respectively. The circumscribed circle of triangle CED and the line through C parallel to AB meet again at point F ($F \neq C$). Suppose that line FD meets segment AB at point G, and let H be the point on line AB such that $\sphericalangle HDA = \sphericalangle GEB$ and $H - A - B$. Given that $DG = EH$, prove that the segments AD and BE intersect on the bisector of angle ACB.

(Miloš Milosavljević)

3. Two players play the following game. They alternate writing integers greater than 1, and a player in turn may not write a number which is a linear combination of numbers written before with nonnegative integer coefficients. The player who cannot perform a move loses the game. Which player, if any, has a winning strategy?

(journal "Kvant" / Aleksandar Ilić)

Time allowed: 270 minutes.
Each problem is worth 7 points.

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Novi Sad – April 6, 2014

Second Day

4. We call a natural number n *nutty* if there exist natural numbers $a > 1$ and $b > 1$ such that $n = a^b + b$. Do there exist 2014 consecutive natural numbers, exactly 2012 of which are nutty? *(Miloš Milosavljević)*

5. A regular n-gon is divided into triangles with $n - 3$ diagonals having no common interior points. At most how many incongruent triangles can occur among these triangles? *(Dušan Djukić)*

6. In a triangle ABC, the internal bisectors at A and B meet the opposite sides at D and E respectively. A rhombus with the non-obtuse angle φ is inscribed in the quadrilateral $ABDE$, with a vertex on each side of $ABDE$. If $\sphericalangle BAC = \alpha$ and $\sphericalangle ABC = \beta$, prove that $\varphi \leqslant \max(\alpha, \beta)$.
(Dušan Djukić, IMO 2013 Shortlist)

Time allowed: 270 minutes.
Each problem is worth 7 points.