Serbian Mathematical Olympiad 2014

for high school students

Novi Sad, April 5–6

Problems and Solutions

Edited by Dušan Djukić

Cover photo: Poganovo Monastery, near Pirot

SHORT HISTORY AND SYSTEM

Mathematical competitions in Serbia have been held since 1958. In the first years only republic competitions within the former Yugoslavia, which Serbia was a part of, were held. The first Federal Mathematical Competition in Yugoslavia was held in Belgrade in 1960, and since then it was held regularly every year, skipping only 1999 for a non-mathematical reason. The system has undergone relatively few changes. The earliest Federal Competitions were organized for 3rd and 4th grades of high school only; 2nd grade was added in 1970, and 1st grade in 1974. Since 1982, 3rd and 4th grades compose a single category. After the breakdown of the old Yugoslavia in 1991, the entire system was continued in the newly formed FR Yugoslavia, later renamed Serbia and Montenegro. The separation of Montenegro finally made the federal competition senseless as such. Thus, starting with 2007, the federal competition and team selection exam are replaced by a two-day Serbian Mathematical Olympiad.

Today a mathematical competition season in Serbia consists of four rounds of increasing difficulty:

- Municipal round, held in early February. The contest consists of 5 problems, each 20 points worth, to be solved in 3 hours. Students who perform well qualify for the next round (50-60 points are usually enough).

- Regional round, held in late February in the same format as the municipal round. Each student's score is added to that from the municipal round. Although the number of students qualified for the state round is bounded by regional quotas, a total score of 110-120 should suffice.

- State (republic) round, held in late March in a selected town in the country. There are roughly 200 to 300 participants. The contest consists of 5 problems in 4 hours.

- Serbian Mathematical Olympiad (SMO), held in early April in a selected place in the country. The participants are selected through the state round: 28 from A category (distribution among grades: 4+8+8+8), 3 from B category (0+0+1+2), plus those members of the last year's olympic team who did not manage to qualify otherwise. Six most successful contestants are invited to the olympic team.

Since 1998, contests for each grade on the three preliminary rounds are divided into categories A (specialized schools and classes) and B (others). A student from category B is normally allowed to work the problems for category A instead. On the SMO, all participants work on the same problems.

The 8-th Serbian Mathematical Olympiad for high school students took place in Novi Sad on April 5–6, 2014. There were 36 students from Serbia and 7 guest students from Russia. The average score on the contest was 12.36 points. Problems 1, 2 and 4 turned out to be relatively easy, problem 5 medium, while problems 3 and 6 were difficult and no student solved them (apart from a Russian student on problem 3).

The team for the 31-st Balkan MO and 55-th IMO was selected based on the contest:

Žarko Randjelović	HS "Svetozar Marković", Niš	29 points
Dušan Drobnjak	Math High School, Belgrade	28 points
Andjela Šarković	HS "Svetozar Marković", Niš	28 points
Maksim Stokić	Math High School, Belgrade	26 points
Luka Vukelić	Math High School, Belgrade	21 points
Ivan Tanasijević	Math High School, Belgrade	21 points

In this booklet we present the problems and full solutions of the Serbian Mathematical Olympiad and the Balkan Mathematical Olympiad.

Serbian MO 2014 – Problem Selection Committee

- Vladimir Baltić
- Bojan Bašić
- Dušan Djukić
- Miljan Knežević
- Miloš Milosavljević
- Marko Radovanović *(chairman)*
- Miloš Stojaković
- Boris Šobot

SERBIAN MATHEMATICAL OLYMPIAD

for high school students

Novi Sad , 05.04.2014.

First Day

1. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$

$$f\big(xf(y) - yf(x)\big) = f(xy) - xy.$$

(Dušan Djukić)

2. Let D and E be points on sides BC and AC of a triangle ABC, respectively. The circumscribed circle of triangle CED and the line through C parallel to AB meet again at point F ($F \neq C$). Suppose that line FD meets segment AB at point G, and let H be the point on line AB such that $\sphericalangle HDA = \sphericalangle GEB$ and $H - A - B$. Given that $DG = EH$, prove that the segments AD and BE intersect on the bisector of angle ACB. *(Miloš Milosavljević)*

3. Two players play the following game. They alternate writing integers greater than 1, and a player in turn cannot write a number which is a linear combination of numbers written before with nonnegative integer coefficients. The player who cannot perform a move loses the game. Which player, if any, has a winning strategy?
(journal "Kvant" / Aleksandar Ilić)

Time allowed: 270 minutes.
Each problem is worth 7 points.

SERBIAN MATHEMATICAL OLYMPIAD

for high school students

Novi Sad, 06.04.2013.

Second Day

4. We call a natural number n *nutty* if there exist natural numbers $a > 1$ and $b > 1$ such that $n = a^b + b$. Do there exist 2014 consecutive natural numbers, exactly 2012 of which are nutty? *(Miloš Milosavljević)*

5. A regular n-gon is divided into triangles with $n-3$ diagonals having no common interior points. At most how many incongruent triangles can occur among these triangles? *(Dušan Djukić)*

6. In a triangle ABC, the internal bisectors at A and B meet the opposite sides at D and E respectively. A rhombus with the non-obtuse angle φ is inscribed in the quadrilateral $ABDE$, with a vertex on each side of $ABDE$. If $\sphericalangle BAC = \alpha$ and $\sphericalangle ABC = \beta$, prove that $\varphi \leqslant \max(\alpha, \beta)$. *(Dušan Djukić, IMO 2013 Shortlist)*

Time allowed: 270 minutes.
Each problem is worth 7 points.

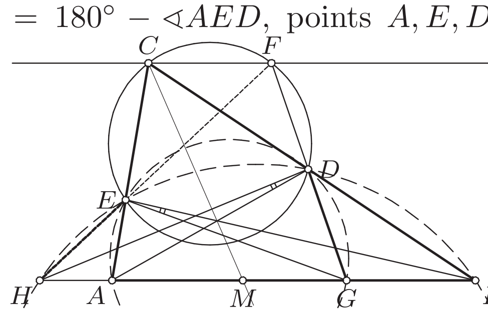
SOLUTIONS

1. Substituting $y = 0$ yields $f(xf(0)) = f(0)$. If $f(0) \neq 0$, the expression $xf(0)$ takes all real values, so f is a constant function, which cannot be a solution. Therefore $f(0) = 0$.

Setting $y = x$ gives us $f(0) = f(x^2) - x^2$, i.e. $f(x^2) = x^2$. Thus $f(x) = x$ for all $x \geqslant 0$. Next, we set $x, y < 0$. In this case $f(xy) = xy$, so we have $f(xf(y) - yf(x)) = 0$, which can only hold if $xf(y) - yf(x) \leqslant 0$. Analogously, we also have $yf(x) - xf(y) \leqslant 0$, implying $yf(x) = xf(y)$, so $f(x)/x = f(y)/y$. It follows that $f(x) = cx$ for all $x < 0$, where c is some constant. Now for $x < 0 < y$ we have $f((1-c)xy) = f(xy) - xy = (c-1)xy$, i.e. $f(z) = -z$ for $z = (1-c)xy$. If $c = 1$, we get the obvious solution $f(x) \equiv x$. On the other hand, if $c \neq 1$, we have $z \neq 0$ and hence $f(z) \in \{cz, z\}$, so we must have $c = -1$, and we conclude that $f(x) = |x|$ for all x. This function also is a solution: All cases except $x > 0 > y$ have already been verified, while for $x > 0 > y$ we have $-2xy = f(-2xy) = f(xf(y) - yf(x)) = f(xy) - xy = -2xy$.

Therefore the only solutions are $f(x) = x$ and $f(x) = |x|$.

2. Since $\sphericalangle AGD = 180° - \sphericalangle CFD = \sphericalangle CED = 180° - \sphericalangle AED$, points A, E, D, G are concyclic, which gives us $\sphericalangle DAG = \sphericalangle DEG$. This now implies $\sphericalangle DHB = \sphericalangle DAG - \sphericalangle HDA = \sphericalangle DEG - \sphericalangle BEG = \sphericalangle DEB$, so points H, E, D, B are concyclic as well. It follows that $\sphericalangle BHE = 180° - \sphericalangle BDE = \sphericalangle CDE = \sphericalangle CFE$, which means that points F, E and H are collinear.



Now, since $EH = EF \cdot \frac{AE}{EC}$ and $DG = DF \cdot \frac{DB}{CD}$, the condition $EH = DG$ becomes $\frac{AE}{EC} \cdot \frac{CD}{DB} = \frac{DF}{EF}$. On the other hand, $\sphericalangle DEF = \sphericalangle DCF = \sphericalangle ABC$ and $\sphericalangle DFE = \sphericalangle ACB$ imply that $\triangle DEF \sim \triangle ABC$ and hence $\frac{DF}{EF} = \frac{AC}{CB} = \frac{AM}{MB}$, where M is the intersection point of the bisector of angle ACB with side AB. Therefore $\frac{AE}{EC} \cdot \frac{CD}{DB} \cdot \frac{BM}{MA} = 1$, and lines AD and BE meet on CM by the Ceva theorem.

3. For $a, b \in \mathbb{N}$ we define $\mathcal{L}\{a, b\} = \{ax + by \mid x, y \in \mathbb{N}_0\}$. We start with an auxilliary statement.

Lemma. Let $a > 1$ and $b > 1$ be coprime integers.

(a) $N = ab - a - b$ is the largest natural number outside $\mathcal{L}\{a, b\}$.

(b) For $z \in \mathbb{Z}$ we have $z \in \mathcal{L}\{a, b\}$ if and only if $N - z \notin \mathcal{L}\{a, b\}$.

Proof. (a) If $N = (b-1)a - b = ax + by$ for some $x \in \{0, \dots, b-1\}$ and $y \in \mathbb{Z}$, then $x \equiv b - 1 \pmod{b}$, so $x \geqslant b - 1$ and therefore $y < 0$; thus $N \notin \mathcal{L}\{a, b\}$.

(b) Clearly $z \in \mathcal{L}\{a, b\}$ implies $N - z \notin \mathcal{L}\{a, b\}$ (for otherwise $N = z + (N - z) \in \mathcal{L}\{a, b\}$). Now consider some $z \in \mathbb{Z} \setminus \mathcal{L}\{a, b\}$. If $x \in \{0, \dots, b-1\}$ is such that $ax \equiv z \pmod{b}$, then $z < ax$, so $ax - by = z$ for some $y \in \mathbb{N}$. Then we have $N - x = (b - 1 - x)a + b(y - 1) \in \mathcal{L}\{a, b\}$. □

Let player A open the game by writing down a prime number $a \geqslant 5$, followed by player B who writes a number b such that $a \nmid b$. The game is finite, as there are only finitely many natural numbers not belonging to $\mathcal{L}\{a, b\}$. Consequently, one of the players has a winning strategy.

Consider a game in which A chooses number N in his second move. If it is a winning move, A will naturally write N. Suppose otherwise and assume B's winning response c leaves A in a losing position. Then A can choose number c in his second move instead of N and continue following the other player's winning strategy. Indeed, by the Lemma, number $N - c$ is forbidden, so after writing number c number $N = (N - c) + c$ is forbidden as well, so B is now in the same losing position.

4. First we observe that there exist 2012 consecutive numbers all of which are nutty: it suffices to take $N + 2, N + 3, \dots, N + 2013$, where $N = 2^{2013!}$.

For a natural number n, denote by $f(n)$ the number of nutty numbers among $n, n + 1, \dots, n + 2013$. Since $f(1) < 2012$ (numbers $1, 2, 3, 4, 5$ are not nutty), $f(N) \geqslant 2012$ and $|f(n+1) - f(n)| \leqslant 1$ for each n, there exists n with $f(n) = 2012$.

Second solution. Denote $N = \frac{2014!}{2011}$. Numbers $2^N + i$ are nutty for $2 \leqslant i \leqslant 2010$ and $2012 \leqslant i \leqslant 2014$; let us prove that $2^N + 1$ and $2^N + 2011$ are not.

Suppose that $2^N + 2011 = a^b + b$ (number $2^N + 1$ is dealt with in a similar way). From $2^b \leqslant a^b < 2^N$ we deduce that $b < N$. We also have $b > 2011$, for otherwise $2011 - b = a^b - 2^N = (a - 2^{\frac{N}{b}})(\dots + 2^{\frac{(b-1)N}{b}}) > 2^{\frac{N}{b}} > 2011$. Next, if $2 \mid a$, then $2^b \mid 2^N - a^b = b - 2011 < 2^b$, which is impossible. Finally, if $2 \nmid a$ and $2 \mid b$, then we have $b - 2011 = (2^{\frac{N}{2}} - a^{\frac{b}{2}})(2^{\frac{N}{2}} + a^{\frac{b}{2}}) > 2^{\frac{N}{2}}$, so $2^N + 2011 = a^b + b > 2^{2^{N/2}}$, impossible again because $2^{\frac{N}{2}} > N$.

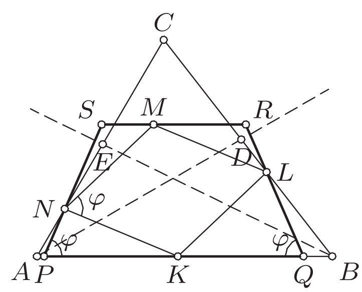
5. The answer is $[\frac{3n-7}{4}]$ for $n > 3$, and 1 for $n = 3$.

Fro $n > 3$, we call a triangle in a triangulation *external, thin* or *fat* if the sides it shares with the polygon are two, one, or none, respectively. Let there be a fat, b thin and c external triangles. The number of sides of the n-gon they cover is $b + 2c = n$. On the other hand, there are $n - 2 = a + b + c$ triangles in total, so these two relations give us $c = a + 2$.

Among triangles that are not fat, there can be at most $[\frac{n-1}{2}]$ incongruent ones, so the total number N of incongruent triangles is at most $\frac{n-1}{2}+a$. On the other hand, there are also $a+2$ external triangles, all of which are congruent, so $N \leqslant n-2-(a+1)=n-a-3$. Addition yields $2N \leqslant (\frac{n-1}{2}+a)+(n-a-3)=\frac{3n-7}{2}$, i.e. $N \leqslant [\frac{3n-7}{4}]$.

Finally, note that drawing diagonals A_0A_{2i} and $A_{2i-2}A_{2i}$ $(1 \leqslant i \leqslant [\frac{n}{4}])$ and A_0A_j $(2[\frac{n}{4}] < j \leqslant n-2)$ yields a triangulation with $[\frac{n-1}{2}]+[\frac{n}{4}]-1=[\frac{3n-7}{4}]$ incongruent triangles.

6. Let $KLMN$ be a rhombus with $K \in AB$, $L \in BD$ and $N \in EA$, and consider the trapezoid $PQRS$ with $PQ \parallel RS$, $P,Q \in AB$, K,L,M,N on PQ,QR,RS,SP respectively, and $\angle PQR = \angle QPS = \varphi$. Suppose that $\varphi > \alpha, \beta$. Then both R and S lie outside $\triangle ABC$.



Since $\angle SNM = 180^\circ - \varphi - \angle NMS = \angle RML$ and $\angle LRM = \angle MSN$, triangles LRM and MSN are congruent and hence $LR = MS$. Similarly, if Q' is the point on AB such that $\angle LQ'B = \varphi$, triangles LRM and $KQ'L$ are congruent, so $MR = LQ' = LQ$. Therefore $RQ = RL + LQ = SM + MR = RS$.

Now $d(R, AB) = RQ \sin\varphi > RS \sin\alpha > d(R, AC)$ and analogously $d(S, AB) > d(S, BC)$, which means that R and S lie above the angle bisectors AD and BE respectively (i.e. in the same half-planes as C), and hence both lie above the line DE. This is a contradiction because the segments RS and DE meet at M.

Second solution. By $d(X,p)$ we denote the distance from point X to line p.

Lemma. If M is a point on segment DE, then $d(M,AB) = d(M,AC) + d(M,BC)$.

Proof. Denoting $\frac{DM}{DE} = k$, we have $d(M,AB) = kd(E,AB) + (1-k)d(D,AB)$
$= kd(E,BC) + (1-k)d(D,AC) = d(M,BC) + d(M,AC)$. □

We use the same notation as in the first solution. Let a be the side length of the rhombus and O be its center. We have

$$d(M,AC)+d(M,BC) = a(\sin\sphericalangle MNC + \sin\sphericalangle MLC),$$
$$\text{and} \quad d(M,AB) = 2d(O,AB) = d(L,AB)+d(N,AB) = a(\sin\sphericalangle NKA + \sin\sphericalangle LKB).$$

However, if $\varphi > \alpha, \beta$, then $\sphericalangle NKA = \sphericalangle MNC + \varphi - \alpha > \sphericalangle MNC$ and similarly $\sphericalangle LKB > \sphericalangle MLC$, so the above equalities imply $d(M,AB) > d(M,AC) + d(M,BC)$, a contradiction.

Remark. The equality is attained for any triangle ABC with $CA = CB$ and any rhombus inscribed in $ABDE$.

The 31-st Balkan Mathematical Olympiad was held from May 2 to May 7 in Pleven in Bulgaria. The results of the Serbian contestants are given in the following table:

	1	2	3	4	Total	
Žarko Randjelović	10	10	10	10	40	Gold medal
Dušan Drobnjak	10	10	10	10	40	Gold medal
Andjela Šarković	4	10	0	9	23	Bronze medal
Maksim Stokić	10	10	10	0	30	Bronze medal
Ivan Tanasijević	10	10	10	10	40	Gold medal
Luka Vukelić	10	2	1	1	14	H. mention

Unlike most other years, this year the contest lacked harder problems, so the full score was required for a gold medal. Thus, 9 contestants (8 officiall + 1 guest) with 40 points were awarded gold medals, 19 (13+6) with 33-39 points were awarded silver medals, and 44 (18+26) with 19-32 points were awarded bronze medals.

The (unofficial) team ranking is given below:

Member Countries		Guest Teams	
1. Turkey	214	Kazakhstan	181
2. Bulgaria	211	Italy	159
3. Romania	209	Bulgaria B	153
4. Serbia	187	United Kingdom	133
5. Greece	182	Tajikistan	114
6. Moldova	147	Azerbaijan	109
7. Macedonia (FYR)	101	Saudi Arabia	100
8. Cyprus	99	Turkmenistan	99
9. Albania	95	Kyrgyzstan	45
10. Montenegro	48	Uzbekistan	39

BALKAN MATHEMATICAL OLYMPIAD

Pleven, Bulgaria , 04.05.2014.

1. Let x, y and z be positive real numbers satisfying $xy + yz + zx = 3xyz$. Prove that

$$x^2y + y^2z + z^2x \geqslant 2(x + y + z) - 3$$

and determine when equality holds. *(United Kingdom)*

2. A *special number* is a positive integer n for which there exist positive integers a, b, c and d with

$$n = \frac{a^3 + 2b^3}{c^3 + 2d^3}.$$

Prove that:

(a) there are infinitely many special numbers;

(b) 2014 is not a special number. *(Romania)*

3. Let $ABCD$ be a trapezoid inscribed in a circle Γ with diameter AB. Its diagonals AC and BD intersect at point E. The circle with center B and radius BE meets Γ at points K and L, where K is on the same side of AB as C. The line perpendicular to BD at point E intersects line CD at point M. Prove that lines KM and DL are perpendicular. *(Greece)*

4. Let n be a positive integer. A regular hexagon with side n is divided into equilateral triangles with side 1 by lines parallel to its sides. Find the number of regular hexagons all of whose vertices are among the vertices of equilateral triangles. *(United Kingdom)*

Time allowed: 270 minutes.
Each problem is worth 10 points.

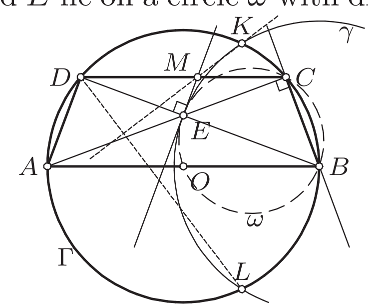
SOLUTIONS

1. The problem condition is equivalent to $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=3$. By the mean inequality it holds that $x^2y+\frac{1}{y}\geqslant 2\sqrt{x^2y\cdot\frac{1}{y}}=2x$ and analogously $y^2z+\frac{1}{z}\geqslant 2y$ and $z^2x+\frac{1}{x}\geqslant 2z$. Adding these inequalities gives the desired inequality.
Equality holds when $x^2y=\frac{1}{y}$, $y^2z=\frac{1}{z}$ and $z^2x=\frac{1}{x}$, which implies $x=y=z=1$.

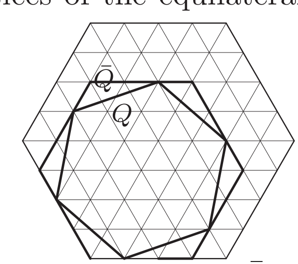
2. (a) Setting $a=nc$ and $b=nd$ we see that $\frac{a^3+2b^3}{c^3+2d^3}=n^3$ is special for all $n\in\mathbb{N}$.

(b) Suppose that $2014=\frac{a^3+2b^3}{c^3+2d^3}$, i.e. $a^3+2b^3=2\cdot 19\cdot 53(c^3+2d^3)$. We can assume w.l.o.g. that $\gcd(a,b,c,d)=1$. The possible residues of cubes modulo 19 are $0,\pm1,\pm7,\pm8$, so $a^3\equiv 2b^3$ can hold only if $19\mid a,b$. However, this implies $19^3\mid a^3+2b^3$ and thus $19^2\mid c^3+2d^3$, so we also deduce $19\mid c,d$, a contradiction.

3. Let O be the center of circle Γ. Points O,B,C and E lie on a circle ω with diameter BE which is tangent to line EM and circle $\gamma(B,BE)$. The radical axes of pairs of circles (Γ,γ), (Γ,ω), (γ,ω) are lines KL, BC i EM, so they are concurrent or parallel. Hence KL is the perpendicular bisector of CM, so $KM=KC$. Now $\sphericalangle KMC+\sphericalangle LDC=\sphericalangle KCM+\sphericalangle LDB+\sphericalangle BDC=\sphericalangle KCD+\sphericalangle KDB+\sphericalangle BDC=\sphericalangle KCD+\sphericalangle KDC+2\sphericalangle BDC=\sphericalangle CBD+\sphericalangle BEC=90^\circ$, and hence $KM\perp DL$.



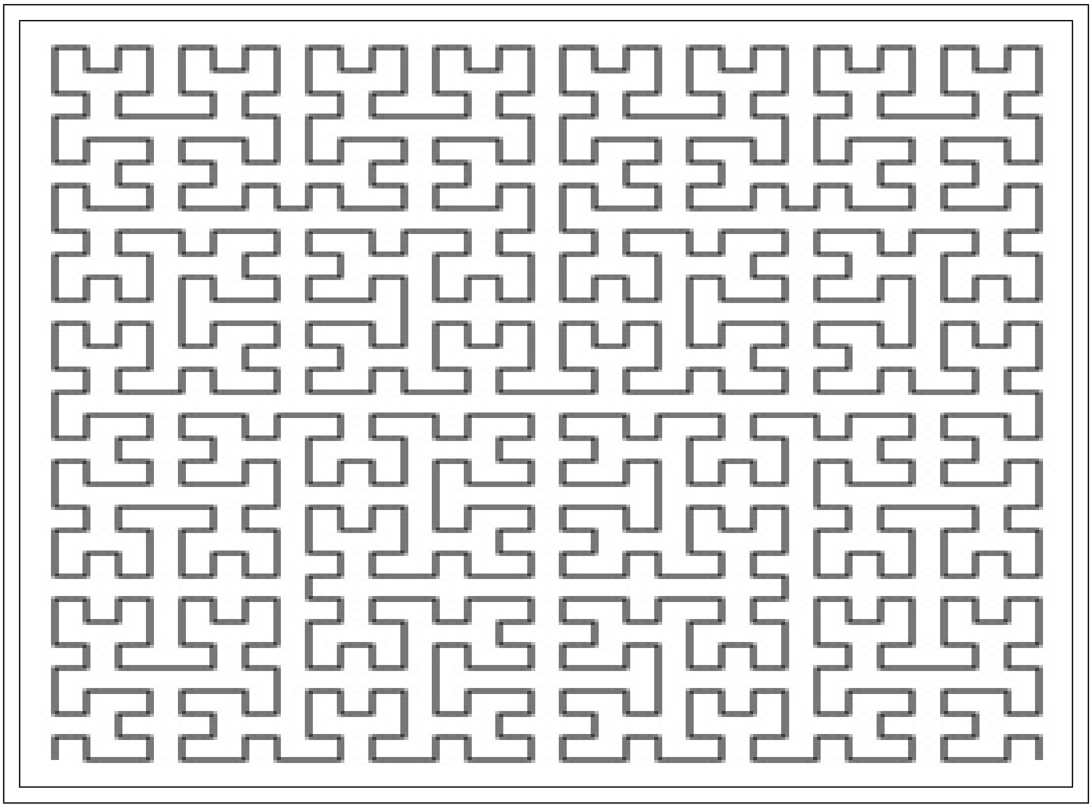
4. Denote the given hexagon by P. We call the vertices of the equilateral triangles simply *nodes*. For every regular hexagon Q with the vertices in nodes, consider the hexagon $\bar{Q}$ circumscribed about Q with sides parallel to the sides of P. The vertices of $\bar{Q}$ are nodes as well.



For $0\leqslant m<n$, hexagon $\bar{Q}$ of side $n-m$ can be chosen in $3m^2+3m+1=(m+1)^3-m^3$ ways. Also, given $\bar{Q}$, hexagon Q can be chosen in $n-m$ ways. Thus the total number of hexagons Q equals

$$\sum_{m=0}^{n-1}(n-m)((m+1)^3-m^3)=\sum_{m=0}^{n-1}(n-m)(m+1)^3-\sum_{m=0}^{n-1}(n-m)m^3$$
$$=\sum_{m=1}^{n}(n-m+1)m^3-\sum_{m=0}^{n-1}(n-m)m^3=\sum_{m=1}^{n}m^3=\frac{n^2(n+1)^2}{4}.$$

The SMO was sponsored by



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