# The $6^{\text{th}}$ Romanian Master of Mathematics Competition

Day 1: Friday, March 1, 2013, Bucharest

Language: English

**Problem 1.** For a positive integer $a$, define a sequence of integers $x_1, x_2, \ldots$ by letting $x_1 = a$ and $x_{n+1} = 2x_n + 1$ for $n \geq 1$. Let $y_n = 2^{x_n} - 1$. Determine the largest possible $k$ such that, for some positive integer $a$, the numbers $y_1, \ldots, y_k$ are all prime.

**Problem 2.** Does there exist a pair $(g, h)$ of functions $g, h\colon \mathbb{R} \to \mathbb{R}$ such that the only function $f\colon \mathbb{R} \to \mathbb{R}$ satisfying $f(g(x)) = g(f(x))$ and $f(h(x)) = h(f(x))$ for all $x \in \mathbb{R}$ is the identity function $f(x) \equiv x$?

**Problem 3.** Let $ABCD$ be a quadrilateral inscribed in a circle $\omega$. The lines $AB$ and $CD$ meet at $P$, the lines $AD$ and $BC$ meet at $Q$, and the diagonals $AC$ and $BD$ meet at $R$. Let $M$ be the midpoint of the segment $PQ$, and let $K$ be the common point of the segment $MR$ and the circle $\omega$. Prove that the circumcircle of the triangle $KPQ$ and $\omega$ are tangent to one another.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.

# The 6th Romanian Master of Mathematics Competition

Day 2: Saturday, March 2, 2012, Bucharest

Language: English

**Problem 4.** Suppose two convex quadrangles in the plane, $P$ and $P'$, share a point $O$ such that, for every line $\ell$ through $O$, the segment along which $\ell$ and $P$ meet is longer than the segment along which $\ell$ and $P'$ meet. Is it possible that the ratio of the area of $P'$ to the area of $P$ be greater than 1.9?

**Problem 5.** Given a positive integer $k \geq 2$, set $a_1 = 1$ and, for every integer $n \geq 2$, let $a_n$ be the smallest solution of the equation

$$x = 1 + \sum_{i=1}^{n-1} \left\lfloor \sqrt[k]{\frac{x}{a_i}} \right\rfloor$$

that exceeds $a_{n-1}$. Prove that all primes are among the terms of the sequence $a_1, a_2, \ldots$.

**Problem 6.** A token is placed at each vertex of a regular $2n$-gon. A *move* consists in choosing an edge of the $2n$-gon and swapping the two tokens placed at the endpoints of that edge. After a finite number of moves have been performed, it turns out that every two tokens have been swapped exactly once. Prove that some edge has never been chosen.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.