International Mathematical

ARHIMEDE

Contest – 7^{th} Edition

Bucharest – 24th-29th June 2013

Problems

1 *Show that in any set of three distinct integers there are two of them say a and b such that the number*

$$a^5b^3 - a^3b^5$$

is a multiple of 10.

2 *For all positive integer n we consider the number $a_n = 4^{6^n} + 1943$. Prove that a_n is divisible by* 2013 *for all $n \geq 1$, and find all values of n for which $a_n - 207$ is the cube of a positive integer.*

3 *Let ABC be a triangle with $\angle ABC = 120°$ and triangle bisectors $(AA_1), (BB_1)$, (CC_1), respectively. $B_1F \perp A_1C_1$, where $F \in (A_1C_1)$. Let R, I and S be the centers of circles which are inscribed in triangles $C_1B_1F, C_1B_1A_1, A_1B_1F$, and $B_1S \cap A_1C_1 = \{Q\}$. Show that R, I, S, Q are on the same circle.*

4 *Let p, n be positive integers such that p is prime and $p < n$. If p divides $n + 1$ and $\left(\left[\frac{n}{p}\right], (p-1)!\right) = 1$, then prove that $p \cdot \left[\frac{n}{p}\right]^2$ divides $\binom{n}{p} - \left[\frac{n}{p}\right]$. (Here $[x]$ represents the integer part of the real number x.)*

5 *Let Γ be the circumcircle of a triangle ABC and let E and F be the intersections of the bisectors of $\angle ABC$ and $\angle ACB$ with Γ. If EF is tangent to the incircle γ of $\triangle ABC$, then find the value of $\angle BAC$.*

6 *Let p be an odd positive integer. Find all values of the natural numbers $n \geq 2$ for which holds*

$$\sum_{i=1}^{n} \prod_{j \neq i} (x_i - x_j)^p \geq 0,$$

where $x_1, x_2, \ldots, x_n$ are any real numbers.

Solutions

1 *Show that in any set of three distinct integers there are two of them say a and b such that the number*

$$a^5b^3 - a^3b^5$$

is a multiple of 10.

(José Luis Díaz-Barrero, Spain)

Solution. First we observe that the statement holds if the set includes $a = 0$ or $b = 0$. Let us denote by $N(a,b) = a^5b^3 - a^3b^5$. Since $N(-a,-b) = N(a,b)$ and $N(-a,b) = N(a,-b) = -N(a,b)$, then WLOG we may assume that the 3 distinct integers are all positive. Now, it easy to check that $a^5b^3 - a^3b^5$ is even and it is suffice to prove that $N(a,b)$ is a multiple of 5, which will certainly occur if either a or b is multiple of 5. Since

$$N(a,b) = a^3b^3(a^2 - b^2) = a^3b^3(a-b)(a+b),$$

what we have to prove is the following claim:

Given any 3 *positive integers none of which is multiple of* 5, *the sum or difference of* 2 *of them is a multiple of* 5.

Indeed, the last digit of any number not multiple of 5 lie in the set $\{1,2,3,4,6,7,8,9\}$. Let $A = \{1,4,6,9\}$ and $B = \{2,3,7,8\}$ (pigeonholes). Of the 3 integers (pigeons) in our set, by the PHP, at least 2 belong to A or at least 2 belong to B, In any case, either their sum or their difference is a multiple of 5 as can be easily check, and we are done.

□

2 *For all positive integer n we consider the number $a_n = 4^{6^n} + 1943$. Prove that a_n is divisible by* 2013 *for all $n \geq 1$, and find all values of n for which $a_n - 207$ is the cube of a positive integer.*

(Nicolae Papacu, Romania)

Solution. To prove the first part, we begin observing that $2013 = 3 \cdot 11 \cdot 61 = 33 \cdot 61$. Since $9^5 = 81 \cdot 81 \cdot 9 = (61+20)(61+20) \cdot 9$ and $20 \cdot 20 \cdot 9 = 3600 = 61 \cdot 59 + 1$, then we have $9^5 \equiv 1 \pmod{61}$. Since $4^6 = 4096 = 61 \cdot 67 + 9 \equiv 9 \pmod{61}$ and $6^{n-1} = (5+1)^{n-1} = 5m + 1$, $m \in \mathbb{N}$, then for all $n \geq 1$, holds

$$4^{6^n} = \left(4^6\right)^{6^{n-1}} = \left(4^6\right)^{5m+1} \equiv 9^{5m} \cdot 9 \pmod{61} \equiv 9 \pmod{61}$$

So, $a_n = 4^{6^n} + 1943 \equiv 1952 \pmod{61} \equiv 0 \pmod{61}$ and $61 | a_n$, for all $n \geq 1$.

On the other hand, $a_n = 4^{6^n} - 4 + 1947 = 4\left(4^{6^n - 1} - 1\right) + 33 \cdot 59$. Since $6^n - 1 \equiv 0 \pmod 5$, then $6^n - 1 = 5p$, $p \in \mathbb{N}$. Then, we have

$$\begin{aligned} 4^{6^n-1} - 1 &= 4^{5p} - 1 = \left(4^5\right)^p - 1 = 1024^p - 1 \\ &= (1024-1)(1024^{p-1} + \ldots + 1) = 1023 \cdot q = 33 \cdot 31 \cdot q \end{aligned}$$

and $33|a_n$ the jointly with the preceding yields $2013 = 33 \cdot 61 | a_n$ for all $n \geq 1$.

To solve the second part of the statement, we observe that $a_n - 207 = 4^{6^n} + 1736$ is an even integer, say $2x$ with $x \in \mathbb{N}$. From $4^{6^n} + 1736 = (2x)^3$ follows $2^{2\cdot 6^n - 3} + 217 = x^3$ or $2^{3(4\cdot 6^{n-1}-1)} + 217 = x^3$. Putting $2^{4\cdot 6^{n-1}-1} = y$ in the last equation yields

$$x^3 - y^3 = 217 \Leftrightarrow (x-y)(x^2+xy+y^2) = 217 = 7\cdot 31$$

Since $x - y < x^2 + xy + y^2$, then we have two possibilities

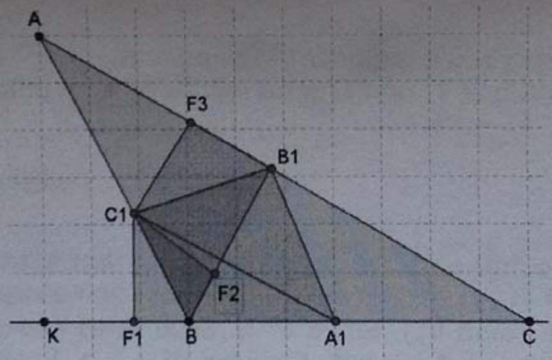
$$\begin{cases} x - y = 1, \\ x^2 + xy + y^2 = 217. \end{cases} \quad \text{or} \quad \begin{cases} x - y = 7, \\ x^2 + xy + y^2 = 31. \end{cases}$$

The solutions of the first system are $(9, 8), (-8, -9)$ and the solutions of the second one $(6, -1)$ and $(1, -6)$. Finally, since $y = 2^{4\cdot 6^{n-1}-1}$ is a positive integer, then $y = 2^{4\cdot 6^{n-1}-1} = 8 = 2^3$ from which follows $n = 1$, and we are done. □

3 *Let ABC be a triangle with $\angle ABC = 120°$ and triangle bisectors $(AA_1), (BB_1)$, (CC_1), respectively. $B_1F \perp A_1C_1$, where $F \in (A_1C_1)$. Let R, I and S be the centers of circles which are inscribed in triangles $C_1B_1F, C_1B_1A_1, A_1B_1F$, and $B_1S \cap A_1C_1 = \{Q\}$. Show that R, I, S, Q are on the same circle.*

(Radu Bairac, Republica Moldova)

Solution. First, we will show that $\angle C_1B_1A_1 = 90°$. Lek $K \in BC$ so that $B \in (KA_1)$, then $\angle ABK = 60°$. Point C_1 is on the bisector of $\angle ACB$ and this implies that $d(C_1, BC) = d(C_1, AC)$ or $C_1F_1 = C_1F_3$, where F_1 is the projection of C_1 on BC and F_3 is the projection of C_1 on AC. Segment BA is the bisector of $\angle KBB_1$ implies that $d(C_1, KB) = d(C_1, BB_1)$ or $C_1F_1 = C_1F_2$, where F_2 is the projection of C_1 on BB_1. So, $C_1F_2 = C_1F_3$ and C_1B_1 is the bisector of $\angle BB_1A$. Let us denote $\angle BB_1C_1 = \alpha$. Likewise, we prove that B_1A_1 is the bisector of $\angle BB_1C$. Let $\angle BB_1A_1 = \angle CB_1A_1 = \beta$. Then, from $\angle AB_1C = 180°$ we have $2\alpha + 2\beta = 180°$ and $\alpha + \beta = 90°$.



Let r_1 be the radius of inscribed circle to $\triangle A_1B_1C_1$, r_2 the radius of inscribed circle to $\triangle C_1B_1F$, and r_3 be the radius of inscribed circle to $\triangle A_1B_1F$, respectively. Considering the properties of right triangles, we have

$$\triangle C_1FB_1 \sim \triangle C_1B_1A_1 \Rightarrow \frac{r_2}{r_1} = \frac{B_1C_1}{C_1A_1} = \cos C_1$$

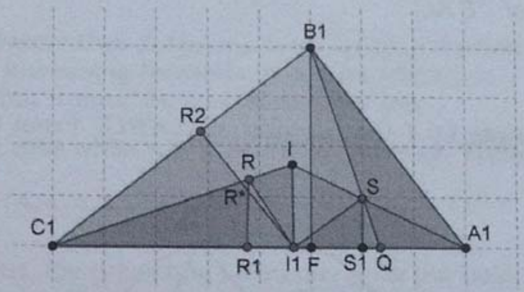
from which follows $r_2 = r_1 \cos C_1 = r_1 \sin A_1$. Likewise,

$$\frac{r_3}{r_1} = \frac{A_1B_1}{C_1A_1} = \cos A_1 \Rightarrow r_3 = r_1 \cos A_1 = r_1 \sin C_1$$

Now, we will see that $I_1R \parallel A_1B_1$ and $I_1S \parallel C_1B_1$, where I_1 is the projection of I on C_1A_1. Let $I_1R_2 \perp C_1B_1$, $R_2 \in (C_1B_1)$ and $I_1R_1 \cap C_1I = \{R^*\}$. On account that $\triangle C_1II_1 \sim \triangle C_1R^+R_2$, then

$$\frac{R^*R_2}{II_1} = \frac{C_1R_2}{C_1I_1} = \cos C_1 \Rightarrow R^*R_2 = r_1 \cos C_1 = r_2$$

from which follows $R^* = RI_1R \perp C_1B_1 \Rightarrow I_1R \parallel A_1B_1$. Likewise, we get $I_1S \parallel C_1B_1$.



In triangle I_1RR_1 we have $r_2 = I_1R \sin A_1 = r_1 \cos C_1 = r_1 \sin A_1$ and $I_1R = r_1$. In triangle I_1SS_1 we have $r_3 = I_1S \sin C_1 = r_1 \cos A_1 = r_1 \sin C_1$ from which follows $I_1S = r_1$. Finally, we get $I_1R = II_1 = I_1S = r_1$. Since $\triangle QSI_1 \sim \triangle QB_1C_1$, then

$$\frac{SI_1}{B_1C_1} = \frac{I_1Q}{C_1Q} \Longrightarrow SI_1 = I_1Q$$

on account that $B_1C_1 = C_1Q$. Now, we can conclude that points R, I, S, and Q lie on the same circle, and we are done.

□

4 *Let p, n be positive integers such that p is prime and $p < n$. If p divides $n+1$ and $\left(\left[\frac{n}{p}\right], (p-1)!\right) = 1$, then prove that $p \cdot \left[\frac{n}{p}\right]^2$ divides $\binom{n}{p} - \left[\frac{n}{p}\right]$. (Here $[x]$ represents the integer part of the real number x.)*

(Diana Alexandrescu, Romania, and José Luis Díaz-Barrero, Spain)

Solution. Since $p \mid n+1$, then $p \mid n+1-p$. So, there exists $k \in \mathbb{N}$ such that $n = kp+p-1$ and $\left[\frac{n}{p}\right] = k$. Now, we have

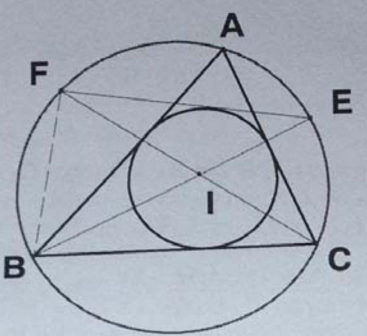
$$\begin{aligned}
\binom{n}{p} - \left[\frac{n}{p}\right] &= \binom{kp+p-1}{p} - k \\
&= \frac{(kp+p-1)(kp+p-2)\ldots(kp+1)(kp)}{p!} - k \\
&= \frac{k(kp+1)(kp+2)\ldots(kp+p-1) - k(p-1)!}{(p-1)!} \\
&= \frac{k(k \cdot p \cdot r + (p-1)!) - k(p-1)!}{(p-1)!} \\
&= \frac{k^2 \cdot p \cdot r}{(p-1)!} \in \mathbb{N}
\end{aligned}$$

Since $\left(\left[\frac{n}{p}\right], (p-1)!\right) = 1$, then $(p-1)!$ divides r and therefore $\binom{n}{p} - \left[\frac{n}{p}\right]$ is divisible by $p \cdot \left[\frac{n}{p}\right]^2$ as we wanted to prove. □

5 *Let Γ be the circumcircle of a triangle ABC and let E and F be the intersections of the bisectors of $\angle ABC$ and $\angle ACB$ with Γ. If EF is tangent to the incircle γ of $\triangle ABC$, then find the value of $\angle BAC$.*

(Iván Gueffner, Spain)

Solution. Let us denote by I the incenter of $\triangle ABC$. From the figure immediately



follows $\angle IBC = \angle IFE$ and $\angle ICB = \angle IEF$. So, $\triangle IBC \simeq \triangle IFE$. Since both have the same height (the radii of the incircle) because BC and EF are both tangent to γ, then $\triangle IBC = \triangle IFE$. Therefore, $IB = IF$. Now, if we denote $\angle ABC = 2\alpha$ and $\angle ACB = 2\beta$, then $\angle BIF = \alpha + \beta$. Now, from isosceles $\triangle IFB$ follows

$$\angle IBF = \angle IFB = \angle CFB = \angle BAC = 180^\circ - 2(\alpha + \beta)$$

Adding up the angles of $\triangle IFB$ yields

$$(\alpha + \beta) + 180^\circ - 2(\alpha + \beta) + 180^\circ - 2(\alpha + \beta) = 180^\circ$$

from which follows $\alpha + \beta = 60^\circ$ and $\angle BAC = 60^\circ$. □

6 *Let p be an odd positive integer. Find all values of the natural numbers $n \geq 2$ for which holds*

$$\sum_{i=1}^{n} \prod_{j \neq i} (x_i - x_j)^p \geq 0,$$

where $x_1, x_2, \ldots, x_n$ are any real numbers.

(Sorin Radulescu, Romania)

Solution. Denote by

$$f_n(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \prod_{j \neq i} (x_i - x_j)^p = (x_1 - x_2)^p (x_1 - x_3)^p \ldots (x_1 - x_n)^p$$

$$+(x_2-x_1)^p(x_2-x_3)^p\dots(x_2-x_n)^p+\dots+(x_n-x_1)^p(x_n-x_2)^p\dots(x_n-x_{n-1})^p$$

Since $f_2(x_1,x_2)=(x_1-x_2)^p+(x_2-x_1)^p=0$ for all $x_1,x_2\in\mathbb{R}$, then for $n=2$ the statement holds and equality occurs. Let $n\geq 3$. Then

$$\begin{aligned} f_n(x_1,x_2,a,a,\dots,a) &= (x_1-x_2)^p(x_1-a)^{p(n-2)}+(x_2-x_1)^p(x_2-a)^{p(n-2)} \\ &= (x_1-x_2)^p\left[(x_1-a)^{p(n-2)}-(x_2-a)^{p(n-2)}\right] \end{aligned}$$

for all $x_1,x_2\in\mathbb{R}$. Observe that $f_n(x_1,x_2,a,a,\dots,a)\geq 0$ when the function $u_a(x)=(x-a)^{p(n-2)}$ be an increasing function. It occurs when $p(n-2)$ is an odd number from which follows that n must be an odd number too.

Now we consider the case when $n\geq 7$ is an odd number. Then, should be

$$f_n(x_1,a,a,a,b,\dots,b)=(x_1-a)^{3p}(x_1-b)^{p(n-4)}\geq 0,$$

for all $x_1,a,b\in\mathbb{R}$. But, the preceding inequality does not hold when $x_1=\dfrac{a+b}{2}$ and $a\neq b$. So, we have to analyze the cases $n=3$ and $n=5$.

(1) For $n=3$ we may suppose WLOG that $x_1\geq x_2\geq x_3$. Let $g(x_1,x_2,x_3)=(x_3-x_1)^p(x_3-x_2)^p$ and $u(x)=(x-x_3)^p$, $x\geq x_3$. Then, we have

(i) u is increasing.

(ii) $g(x_1,x_2,x_3)\geq 0$.

On account of (i) and (ii), we have that

$$\begin{aligned} f(x_1,x_2,x_3) &= (x_1-x_2)^p\left[(x_1-x_3)^p-(x_2-x_3)^p\right]+(x_3-x_1)^p(x_3-x_2)^p \\ &= (x_1-x_2)^p\left[u(x_1)-u(x_2)\right]+g(x_1,x_2,x_3)\geq 0 \end{aligned}$$

and the statement holds for $n=3$.

(2) For $n=5$ we also suppose that $x_1\geq x_2\geq x_3\geq x_4\geq x_5$. Let

$$h(x_1,x_2,x_3,x_4,x_5)=(x_3-x_1)^p(x_3-x_2)^p(x_3-x_4)^p(x_3-x_5)^p$$

and let $v(x)=(x-x_3)^p(x-x_4)^p(x-x_5)^p$, $x\geq x_3$ and $w(x)=(x-x_1)^p(x-x_2)^p(x-x_3)^p$, $x\leq x_3$. Observe that $h(x_1,x_2,x_3,x_4,x_5)\geq 0$ and v and w are increasing. Since

$$\begin{aligned} f(x_1,x_2,x_3,x_4,x_5) &= (x_1-x_2)^p[v(x_1)-v(x_2)] \\ &+(x_4-x_5)^p[w(x_4)-w(x_5)]+h(x_1,x_2,x_3,x_4,x_5), \end{aligned}$$

then $f(x_1,x_2,x_3,x_4,x_5)\geq 0$. We conclude that the natural numbers for which the statement follows are $n=2, n=3$ and $n=5$, respectively.

□