SERBIAN MATHEMATICAL OLYMPIAD

for high school students

Belgrade , 31.03.2012.

First Day

1. Let P be the point on diagonal BD of a parallelogram $ABCD$ such that $\angle PCB = \angle ACD$. The circumcircle of triangle ABD meets the diagonal AC again at point E. Prove that

$$\angle AED = \angle PEB.$$

2. Find all natural numbers a and b such that

$$a \mid b^2, \quad b \mid a^2 \quad \text{and} \quad a+1 \mid b^2+1.$$

3. In some vertices of a square grid 2012×2012 there are a fly and k spiders. In each second, the fly moves to a neighboring vertex or waits, followed by each of the k spiders moving to a neighboring vertex or waiting (there can be more than one spider in the same vertex). At all times, the fly and the spiders know the positions of the others.

a) Find the smallest k such that the spiders can catch the fly in a finite time, no matter the initial positions of the fly and the spiders.

b) Answer the same question for a cube grid $2012 \times 2012 \times 2012$.

(Two vertices are neighboring if they are on a distance 1. A spider catches the fly if they are both at the same vertex.)

Time allowed: 270 minutes.
Each problem is worth 7 points.

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Second Day

4. Find all natural numbers n for which there exists a permutation $(p_1, p_2, \ldots, p_n)$ of numbers $(1, 2, \ldots, n)$ such that the sets $\{p_i + i \mid 1 \le i \le n\}$ and $\{p_i - i \mid 1 \le i \le n\}$ form complete sets of residues modulo n.

5. Let $\mathcal{K}$ be the set of points in the plane with integer coordinates. Does there exist a bijection $f : \mathbb{N} \to \mathcal{K}$ such that for all $a, b, c \in \mathbb{N}$

$$\gcd(a,b,c) > 1 \quad \Longrightarrow \quad f(a),\ f(b),\ f(c) \quad \text{are not collinear ?}$$

6. A train consists of $n > 1$ waggons with gold coins. Some coins are genuine and some are fake, although they all look the same and can only be distinguished by mass: all genuine coins have the same mass, and so do all fake ones, where the two masses differ. The mass of a genuine coin is known. Each waggon contains only genuine coins or only fake ones.

Find the smallest number of measurements on a digital scale by which one can determine all waggons with fake coins and find the mass of a fake coin.

(It is assumed that from each waggon one can take as many coins as needed.)

Time allowed: 270 minutes.
Each problem is worth 7 points.