Serbian Mathematical Olympiad 2012

for high school students

Belgrade, March 31–April 1, 2012

Problems and Solutions

Cover photo: Meanders of Uvac

SHORT HISTORY AND SYSTEM

Mathematical competitions in Serbia have been held since 1958. In the first years only republic competitions within the former Yugoslavia, which Serbia was a part of, were held. The first Federal Mathematical Competition in Yugoslavia was held in Belgrade in 1960, and since then it was held regularly every year, skipping only 1999 for a non-mathematical reason. The system has undergone relatively few changes. The earliest Federal Competitions were organized for 3rd and 4th grades of high school only; 2nd grade was added in 1970, and 1st grade in 1974. Since 1982, 3rd and 4th grades compose a single category. After the breakdown of the old Yugoslavia in 1991, the entire system was continued in the newly formed FR Yugoslavia, later renamed Serbia and Montenegro. The separation of Montenegro finally made the federal competition senseless as such. Thus, starting with 2007, the federal competition and team selection exam are replaced by a two-day Serbian Mathematical Olympiad.

Today a mathematical competition season in Serbia consists of four rounds of increasing difficulty:

- Municipal round, held in early February. The contest consists of 5 problems, each 20 points worth, to be solved in 3 hours. Students who perform well qualify for the next round (50-60 points are usually enough).
- Regional round, held in late February in the same format as the municipal round. Each student's score is added to that from the municipal round. Although the number of students qualified for the state round is bounded by regional quotas, a total score of 110-120 should suffice.
- State (republic) round, held in late March in a selected town in the country. There are roughly 200 to 300 participants. The contest consists of 5 problems in 4 hours.
- Serbian Mathematical Olympiad (SMO), held in early April in a selected place in the country. The participants are selected through the state round: 26 from A category (distribution among grades: 3+5+8+10), 3 from B category (0+0+1+2), plus those members of the last year's olympic team who did not manage to qualify otherwise. Six most successful contestants are invited to the olympic team.

Since 1998, contests for each grade on the three preliminary rounds are divided into categories A (specialized schools and classes) and B (others). A student from category B is normally allowed to work the problems for category A instead. On the SMO, all participants work on the same problems.

The Serbian Mathematical Olympiad 2012 for high school students took place in Belgrade on March 31 and April 1. There were 30 students from Serbia and 8 guest students from Russia and Croatia. The average score on the contest was 12.92 points and all problems were fully solved by the contestants.

The team for the Balkan MO and IMO was to be selected based on the contest, but one of the students was unable to participate on the IMO. The replacement was chosen on an additional team selection exam. Thus the team(s) of Serbia for the 29-th Balkan MO and 53-rd IMO are:

Teodor von Burg	Math High School, Belgrade	
Rade Špegar	Math High School, Belgrade	
Dušan Šobot	Math High School, Belgrade	
Igor Spasojević	Math High School, Belgrade	
Ivan Tanasijević	Math High School, Belgrade	
Ivan Damnjanović	HS "Bora Stanković", Niš	*Balkan MO only*
Lazar Radičević	Math High School, Belgrade	*IMO only*

In this booklet we present the problems and full solutions of the Serbian Mathematical Olympiad with the Additional Team Selection Exam, and the Balkan Mathematical Olympiad.

SERBIAN MATHEMATICAL OLYMPIAD

for high school students

Belgrade , 31.03.2012.

First Day

1. Let P be the point on diagonal BD of a parallelogram $ABCD$ such that $\angle PCB = \angle ACD$. The circumcircle of triangle ABD meets the diagonal AC again at point E. Prove that $\angle AED = \angle PEB$. *(Marko Djikić)*

2. Find all natural numbers a and b such that

$$a \mid b^2, \quad b \mid a^2 \quad \text{and} \quad a+1 \mid b^2+1.$$

(Dušan Djukić)

3. In some vertices of a square grid 2012×2012 there are a fly and k spiders. In each second, the fly moves to a neighboring vertex or waits, followed by each of the k spiders moving to a neighboring vertex or waiting (there can be more than one spider in the same vertex). At all times, the fly and the spiders know the positions of the others.

a) Find the smallest k such that the spiders can catch the fly in a finite time, no matter the initial positions of the fly and the spiders.

b) Answer the same question for a cube grid $2012 \times 2012 \times 2012$.

(Two vertices are neighboring if they are on a distance 1. A spider catches the fly if they are both at the same vertex.) *(Nikola Milosavljević)*

Time allowed: 270 minutes.
Each problem is worth 7 points.

SERBIAN MATHEMATICAL OLYMPIAD

for high school students

Belgrade, 01.04.2012.

Second Day

4. Find all natural numbers n for which there exists a permutation $(p_1, p_2, \dots, p_n)$ of numbers $(1, 2, \dots, n)$ such that the sets $\{p_i + i \mid 1 \le i \le n\}$ and $\{p_i - i \mid 1 \le i \le n\}$ form complete sets of residues modulo n. *(Marko Djikić)*

5. Let $\mathcal{K}$ be the set of points in the plane with integer coordinates. Does there exist a bijection $f : \mathbb{N} \to \mathcal{K}$ such that for all $a, b, c \in \mathbb{N}$

$$\gcd(a, b, c) > 1 \quad \Longrightarrow \quad f(a),\ f(b),\ f(c) \quad \text{are not collinear?}$$

(Stevan Gajović)

6. A train consists of $n > 1$ waggons with gold coins. Some coins are genuine and some are fake, although they all look the same and can only be distinguished by mass: all genuine coins have the same mass, and so do all fake ones, where the two masses differ. The mass of a genuine coin is known. Each waggon contains only genuine coins or only fake ones.

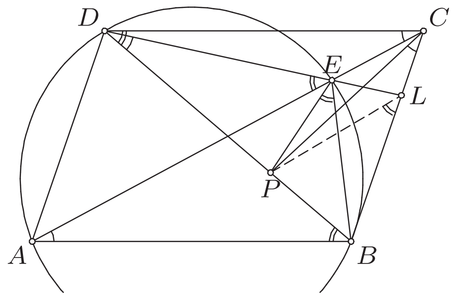
Find the smallest number of measurements on a digital scale by which one can determine all waggons with fake coins and find the mass of a fake coin.

(It is assumed that from each waggon one can take as many coins as needed.) *(Miloš Milosavljević)*

Time allowed: 270 minutes.
Each problem is worth 7 points.

SOLUTIONS

1. We shall assume that $\angle BAC \le 90^\circ$. The other case is analogous. Let the lines DE and BC meet at L. The quadrilateral $CDPL$ is circumscribed because $\angle PDL = \angle PCL$, which gives $\angle PLE = \angle PCD = \angle BCA = \angle DAC = \angle DBE = \angle PBE$, so the quadrilateral $BPEL$ is circumscribed as well. Hence $\angle PEB = \angle PLB = \angle PDC = \angle DBA = \angle DEA$.



Second solution. Let P' be the point on diagonal BD such that $\angle DEA = \angle PEB$. By the sine theorem, $\frac{BP}{DP} = \frac{BP}{CP} \cdot \frac{CP}{DP} = \frac{\sin\angle BCP}{\sin\angle CBD} \cdot \frac{\sin\angle CDB}{\sin\angle PCD}$. Analogously, $\frac{BP'}{DP'} = \frac{\sin\angle BEP'}{\sin\angle EBD} \cdot \frac{\sin\angle EDB}{\sin\angle P'ED}$. Since $\angle BCP = \angle EDB$, $\angle CBD = \angle P'ED$, $\angle CDB = \angle BEP'$ and $\angle PCD = \angle EBD$, we obtain $\frac{BP}{DP} = \frac{BP'}{DP'}$, so $P \equiv P'$.

2. Set $b^2 = ca$. The problem conditions become $b^2 = ca \mid a^4$ and $a+1 \mid ca+1$, which is equivalent to

$$c \mid a^3 \quad \text{and} \quad a+1 \mid c-1.$$

Write $c = d(a+1)+1$, $d \in \mathbb{N}_0$. Since $a^3 \equiv -1 \pmod{a+1}$, we have $\frac{a^3}{c} \equiv -1 \pmod{a+1}$, i.e. $\frac{a^3}{c} = e(a+1)-1$ for some $e \in \mathbb{N}$. It follows that $a^3 = (d(a+1)+1)(e(a+1)-1)$, which after expending and canceling $a+1$ yields $a^2-a+1 = de(a+1)+(e-d)$. Thus $e-d \equiv a^2-a+1 \equiv 3 \pmod{(a+1)}$, so

$$e-d = k(a+1)+3 \quad \text{and} \quad de = a-2-k \quad (k \in \mathbb{Z}).$$

We distinguish the following cases:

(1) $k \notin \{-1,0\}$. Then $de < |e-d|-1$, implying $d=0$. Now $c=1$ and $b^2=a$, so $(a,b)=(t^2,t)$.

(2) $k=-1$. Then $a=d+1$. Now we get $c=a^2$ and $b^2=a^3$, so $(a,b)=(t^2,t^3)$.

(3) $k=0$. We get $a=d^2+3d+2$. Now $c=d(a+1)+1=(d+1)^3$ and $b^2=ca=(d+1)^4(d+2)$. It follows that $d+2=t^2$ for some $t\in\mathbb{N}$, which gives us $(a,b)=\big(t^2(t^2-1),t(t^2-1)^2\big)$, $t\ge 2$.

Thus the possible pairs (a,b) are (t^2,t), (t^2,t^3) and $\big(t^2(t^2-1),t(t^2-1)^2\big)$, $t\in\mathbb{N}$.

3. One spider cannot catch the fly if the fly waits for the spider to arrive to a neighboring cell, and then repeatedly moves onto a cell diagonally adjacent to the spider's. We shall show that the two spiders are enough in both (a) and (b) - denote them by P and Q, the fly by M, and x- and y-coordinates of point A by A_x and A_y.

(a) Set a cartesian coordinate system with the origin at the lower-left corner. First, moving along the x-axis, P can achieve that $P_x = M_x$ in finitely many steps. Analogously, Q achieves that $Q_y = M_y$. From this moment on, the spiders move in the following way: whenever the fly changes its x-coordinate, P does the same to keep $P_x = M_x$, otherwise it moves one step along the y-axis towards the fly; Q moves analogously. In this way, the quantity $|P_y - M_y| + |Q_x - M_x|$ either decreases or does not change, and it can stay unchanged through at most $2 \cdot 2010$ moves (when the fly flees). Hence after finitely many moves at least one summand will be zero, i.e. the fly will be caught.

(b) Disregarding the z-axis, by the part (a), one of the spiders, say P, can achieve that $P_x = M_x$ and $P_y = M_y$. In the next phase, whenever the fly moves along the z-axis, spider P moves towards it, otherwise P moves so as to stay straight under the fly. Clearly, in this manner the fly can make only finitely moves along the z-axis or stay still, before it would be caught by P. Hence, from some moment on, the fly must move within the same xy-plane, without staying still.
Next, spider Q reaches the xy-plane of the fly. Also, staying still if needed, Q achieves that $f = |Q_x - M_x| + |Q_y - M_y|$ be even. In every subsequent move, spider Q moves towards the fly along the x-axis if $|Q_x - M_x| > |Q_y - M_y|$, otherwise it moves along the y-axis. After every move, the quantity f does not increase or change parity, and only finitely many times can it stay unchanged. Therefore, in some moment we shall have $f = 0$ and the fly will be caught.

4. Suppose that such a permutation exists. Since $\{p_i + i \mid 1 \le i \le n\}$ is a complete residue system modulo n, we have $\sum_{k=1}^n k \equiv \sum_{i=1}^n (p_i + i) \equiv \sum_{i=1}^n i + \sum_{i=1}^n p_i \equiv 2\sum_{k=1}^n k \pmod n$, hence $\sum_{k=1}^n k = \frac{n(n+1)}{2} \equiv 0 \pmod n$, which gives us $2 \nmid n$. Moreover, it holds that $2\sum_{k=1}^n k^2 \equiv \sum_{k=1}^n ((p_i+i)^2 + (p_i-i)^2) \equiv \sum_{k=1}^n (2p_i^2 + 2i^2) \equiv 4\sum_{k=1}^n k^2$, so $2\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{3} \equiv 0 \pmod n$, implying $3 \nmid n$. Therefore, we must have $(n, 6) = 1$.

On the other hand, if $(n, 6) = 1$ and $p_i \equiv 2i \pmod n$, $p_i \in \{1, \dots, n\}$, then $(p_1, p_2, \dots, p_n)$ is permutation of $\{1, \dots, n\}$ and satisfies the conditions, because $\{p_i + i \mid 1 \le i \le n\} \equiv \{3i \mid 1 \le i \le n\}$ and $\{p_i - i \mid 1 \le i \le n\} \equiv \{i \mid 1 \le i \le n\}$ $\pmod n$ are complete residue systems modulo n.

5. Arrange all vertices of the grid in a sequence $A_1, A_2, \dots$. We shall construct a desired bijection inductively.
Set $f(1) = A_1$. Suppose that $f(1), \dots, f(n-1)$ have been chosen, and set $f(n)$ to be the point A_m with the smallest m such that, for any $i, j \le n$, $(i, j, n) > 1$, A_m does not lie on line $f(i)f(j)$. There are finitely many lines $f(i)f(j)$, so there is indeed an integer point not lying on any. Observe that, for a prime p, $f(p)$ becomes the unchosen point with the smallest index, which guarantees that every integer point will occur as $f(n)$ for some n.

The bijection f satisfies all problem conditions.

Second solution. If n is composite, set $f(n)$ to be the point (n, n^2). For any prime p, let $f(p)$ be any admitted point on the minimal distance from the origin $(0,0)$. We shall show that f is a well-defined bijection satisfying the conditions.
For each p there is an admitted point - e.g. any unchosen point on the parabola $y = x^2$. Indeed, an arbitrary line through this point meets the parabola in at most one other point, so it contains the image of at most one composite number.
On the other hand, for any arbitrary point $A \in \mathcal{K} \setminus \{(n, n^2) \mid n \text{ is composite}\}$ there is a prime p for which A is an admitted point. Namely, there are only finitely many lines through A containing two integer points on $y = x^2$ (different from A); denote these lines by $p_1, \dots, p_k$ and the intersections of p_i with the parabola $y = x^2$ by A_i, B_i. It suffices to take p not dividing $(f^{-1}(A_i), f^{-1}(B_i))$ for any i.

6. We show that the smallest number of measurements is 2. Denote the masses of a genuine and fake coin by x and y respectively, and let $a_i = 1$ if the i-th waggon contains genuine coins, and $a_i = 0$ otherwise.

In the first measurement, take one coin from each waggon; let m_1 be the result. Then $a_1 + a_2 + \cdots + a_n = \frac{nx - m_1}{x - y}$. We assume that $m_1 \neq nx$, for there would be no fake coins otherwise. In the second measurement, for a suitable $q \in \mathbb{N}$, take q^{i-1} coins from the i-th waggon; let m_2 be the result. We have $a_1 + qa_2 + \cdots + q^{n-1}a_{n-1} = \frac{(1+q+\cdots+q^{n-1})x - m_2}{x-y}$. This yields

$$f(a_1, a_2, \dots, a_n) = \frac{a_1 + qa_2 + \cdots + q^{n-1}a_{n-1}}{a_1 + a_2 + \cdots + a_n} = \frac{(1 + q + \cdots + q^{n-1})x - m_2}{nx - m_1}.$$

If we can choose $q \in \mathbb{N}$ such that the function $f : \{0,1\}^n \setminus \{(0,0,...,0)\} \to \mathbb{R}$ is injective, this will mean that the value of f will uniquely determine $a_1, \dots, a_n$, i.e. the waggons with fake coins, and finally $y = x - \frac{nx - m_1}{a_1 + a_2 + \cdots + a_n}$.

For fixed $\mathfrak{a} = (a_1, a_2, ..., a_n)$ and $\mathfrak{a} = (b_1, b_2, ..., b_n)$, the equality $f(\mathfrak{a}) = f(\mathfrak{b})$ is equivalent to $P_{\mathfrak{a},\mathfrak{b}}(q) = (a_n b - b_n a)q^{n-1} + \cdots + (a_2 b - b_2 a)q + (a_1 b - b_1 a) = 0$, where $a = a_1 + a_2 + ... + a_n \neq 0 \neq b_1 + b_2 + ...b_n = b$. Therefore, if f is not injective, then q is a zero of the polynomial $P(q) = \prod_{\mathfrak{a},\mathfrak{b}} P_{\mathfrak{a},\mathfrak{b}}(q)$. Since the polynomials $P_{\mathfrak{a},\mathfrak{b}}$ are nonzero, there are only finitely many numbers q with $P(q) = 0$, so there exists q for which f is injective, proving that two measurements are enough.

One measurement is not enough. Taking k_i coins from the i-th waggon yields the equation $k_1 a_1 + \cdots + k_n a_n = \frac{kx - m}{x - y}$ (where $k = k_1 + \cdots + k_n$) which does not have a unique solution: for instance, both $(1, 0, \dots, 0, x - \frac{kx-m}{k_1})$ and $(0, 0, \dots, 1, x - \frac{kx-m}{k_n})$ are possible solutions for $(a_1, a_2, \dots, a_n, y)$.

The 29-th Balkan Mathematical Olympiad was held from April 26 to May 2 in Antalya in Turkey. The results of Serbian contestants are given in the following table:

	1	2	3	4	Total	
Teodor von Burg	10	10	10	10	40	Gold Medal
Dušan Šobot	10	10	10	10	40	Gold Medal
Igor Spasojević	10	10	10	10	40	Gold Medal
Rade Špegar	10	10	10	8	38	Silver Medal
Ivan Damnjanović	0	10	10	1	21	Bronze Medal
Ivan Tanasijević	10	10	0	0	20	Bronze Medal

After the contest, 14 contestants (13 officially + 1 unofficially) with 39-40 points were awarded gold medals, 28 (10+18) with 30-38 points were awarded silver medals, and 45 (19+26) with 20-28 points were awarded bronze medals.

The unofficial ranking of the teams is given below:

Member Countries		Guest Teams	
1. Turkey	226	Turkey B	196
2. Romania	213	Italy	172
3. Serbia	199	France	161
4. Bulgaria	162	Kazakhstan	156
5. Greece	154	United Kingdom	139
6. Bosnia-Herzegovina	140	Tajikistan	138
7. Moldova	133	Saudi Arabia	131
8. Macedonia (FYR)	109	Turkmenistan	107
9. Cyprus	79	Azerbaijan	83
10. Albania	41	Indonesia	68
11. Montenegro	16	Afghanistan	0

BALKAN MATHEMATICAL OLYMPIAD

Antalya, Turkey , 28.04.2012.

1. Let A, B and C be points lying on a circle Γ with center O. Assume that $\angle ABC > 90^\circ$. Let D be the point of intersection of the line AB with the line perpendicular to AC at C. Let ℓ be the line through D which is perpendicular to AO. Let E be the point of intersection of ℓ with the line AC, and let F be the point of intersection of Γ with ℓ that lies between D and E. Prove that the circumcircles of triangles BFE and CFD are tangent at F. *(Romania)*

2. Prove that

$$\sum_{\text{cyc}}(x+y)\sqrt{(z+x)(z+y)} \geq 4(xy+yz+zx)$$

for all positive real numbers x, y and z.
Remark. The notation above means that the left-hand side is

$$(x+y)\sqrt{(z+x)(z+y)} + (y+z)\sqrt{(x+y)(x+z)} + (z+x)\sqrt{(y+z)(y+x)}.$$

(Saudi Arabia)

3. Let n be a positive integer. Let $P_n = \{2^n, 2^{n-1}\cdot 3, 2^{n-2}\cdot 3^2, \cdots, 3^n\}$. For each subset X of P_n, we write S_X for the sum of all elements of X, with the convention that $S_\emptyset = 0$ where $\emptyset$ is the empty set. Suppose that y is a real number with $0 \leq y \leq 3^{n+1} - 2^{n+1}$. Prove that there is a subset Y of P_n such that $0 \leq y - S_Y < 2^n$. *(United Kingdom)*

4. Find all functions f from the set of positive integers to itself such that the following conditions both hold:

(i) $f(n!) = f(n)!$ for all positive integers n;

(ii) $m-n$ divides $f(m)-f(n)$ whenever m and n are different positive integers. *(Saudi Arabia)*

Time allowed: 270 minutes.
Each problem is worth 10 points.

SOLUTIONS

1. Let G be the point on Γ diametrically opposite of A. Then E is the orthocenter of $\triangle DAG$, so G lies on line BE. Since $\angle CDF = \angle GAC = \angle GFC$ and $\angle FBE = \angle FAG = \angle GFE$, line FG is the common tangent to circles CFD and BFE at F, hence the two circles are tangent.

2. Since $(z+x)(z+y) \geq (z+\sqrt{xy})^2$ by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \textstyle\sum_{\text{cyc}}(x+y)\sqrt{(z+x)(z+y)} &\geq \textstyle\sum_{\text{cyc}}[(x+y)z+(x+y)\sqrt{xy}] \\ &\geq \textstyle\sum_{\text{cyc}}[(x+y)z+2xy] = 4(xy+yz+zx). \end{aligned}$$

3. We use induction on n, The statement is directly verified for $n=1$; assume that it holds for $n-1$. Consider y with $0 \leq y \leq 3^{n+1}-2^{n+1}$. We distinguish two cases.

(i) $0 \leq y \leq 2\cdot 3^n - 2^{n+1}$. By the inductive hypothesis, there is $Y' \subset P_{n-1}$ such that $0 \leq \frac{y}{2} - S_{Y'} < 2^{n-1}$. We can take $Y = 2Y'$ $(=\{2t \mid t \in Y'\})$.

(ii) $2\cdot 3^n - 2^{n+1} \leq y \leq 3^{n+1}-2^{n+1}$. This means that $0 < 3^n - 2^{n+1} \leq y - 3^n \leq 2\cdot 3^n - 2^{n+1}$, so by the inductive hypothesis there is $Y' \subset P_{n-1}$ such that $0 \leq \frac{y-3^n}{2} - S_{Y'} < 2^{n-1}$. We can take $Y = 2Y' \cup \{3^n\}$.

4. Since $f(1) = f(1)!$ and $f(2) = f(2)!$, we have $f(1), f(2) \in \{1,2\}$.
Suppose that $f(3) = 3$. If we define $n_0 = 3$ and $n_{i+1} = n_i!$ for $i \geq 0$, then $f(n_i) = n_i$ by induction. Let m be an arbitrary natural number. Since $m - n_i$ divides $f(m) - f(n_i)$, it divides $f(m) - m$ for all i, so $f(m) - m$ has infinitely many divisors; hence $f(m) = m$ for all m.
Now suppose that $f(3) \neq 3$. Since $4 = 3! - 2 \mid f(3)! - f(2) = f(3) - 2$, we have $4 \nmid f(3)$, so $f(3) \in \{1,2\}$. Moreover, $n! - 3$ divides $f(n)! - f(3)$ for all $n \geq 4$, implying $3 \nmid f(n)!$ and hence $f(n) \in \{1,2\}$ for all n. Then it is easily seen that f must be constant.
Thus the solutions are $f \equiv 1$, $f \equiv 2$, and $(\forall x) f(x) = x$

Additional IMO Team Selection Exam

Belgrade , 16.05.2012.

1. A polynomial $P(x)$ of degree 2012 with real coefficients is such that the inequality

$$P(a)^3 + P(b)^3 + P(c)^3 \geq 3P(a)P(b)P(c)$$

holds for any real numbers a, b, c with $a+b+c=0$. Can the polynomial $P(x)$ have 2012 different real zeros? *(Miloš Milosavljević)*

2. By $\sigma(x)$ we denote the sum of divisors of a natural number x, including 1 and x. For each $n \in \mathbb{N}$, let $f(n)$ be the number of natural numbers m, $m \leq n$, for which $\sigma(m)$ is odd. Show that there exist infinitely many numbers n such that $f(n) \mid n$. *(Bojan Bašić)*

3. Let P and Q be points inside a triangle ABC such that $\angle PAC = \angle QAB$ and $\angle PBC = \angle QBA$.

a) Prove that the orthogonal projections of P and Q onto the sides of the triangle lie on a circle.

b) Let D and E be the feet of perpendiculars from P to lines BC and AC, and let F be that from Q to AB. The lines DE and AB meet at point M. Prove that MP is perpendicular to CF. *(Dušan Djukić)*

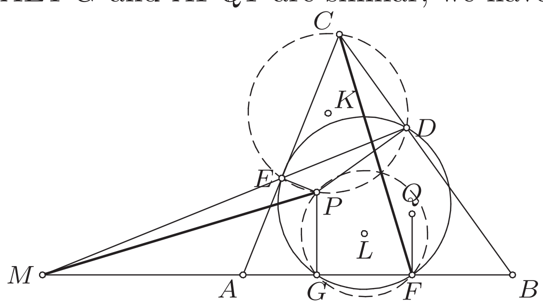
Time allowed: 270 minutes.
Each problem is worth 7 points.

SOLUTIONS

1. Since $x^3+y^3+z^3-3xyz=\frac{1}{2}(x+y+z)\big((x-y)^2+(y-z)^2+(z-x)^2\big)$, the problem condition is equivalent to "$P(a)+P(b)+P(c)\geqslant 0$ whenever $a+b+c=0$".
Consider the polynomial $P(x)=\prod_{k=0}^{2011}(x-1-\frac{k}{4022})$. For $x\leqslant 1$ or $x\geqslant\frac{3}{2}$ it holds that $P(x)\geqslant 0$; Moreover, $P(x)>1$ for $x\leqslant 0$. For $1<x<\frac{3}{2}$, each factor $x-1-\frac{k}{4022}$ is less than $\frac{1}{2}$ in absolute value, hence $P(x)>-\frac{1}{2^{2012}}$. If $a+b+c=0$, then at least one of a,b,c is nonpositive, say $a\leq 0$. Then $P(a)>1$ and $P(b),P(c)>-\frac{1}{2^{2012}}$, so we have $P(a)+P(b)+P(c)>0$. Thus the polynomial $P(x)$ satisfies all conditions.

2. Recall that if $n=p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$ is the factorization of $n\in\mathbb{N}$ into primes, then $\sigma(n)=\prod_{i=1}^{k}(1+p_i+\cdots+p_i^{r_i})$. The number $\sigma(n)$ is off if and only if all factors $1+p_i+\cdots+p_i^{r_i}$ are odd, which is equivalent to $p_i=2$ or $2\mid r_i$. Thus $\sigma(n)$ is odd if and only if either n or $n/2$ is a perfect square, which yields $f(n)=[\sqrt{n}]+[\sqrt{\frac{n}{2}}]$. Note that $f(n)\leqslant f(n+1)$ for each n. Furthermore, the quotient $\frac{n}{f(n)}$ is unbounded, hence for each $k\in\mathbb{N}$ there is the smallest $n=n_k$ for which $\frac{n}{f(n)}\geqslant k$. For $k>1$ we have $n_k>1$ and $\frac{n_k-1}{f(n_k-1)}<k$, which implies $n_k\geqslant kf(n_k)\geqslant kf(n_k-1)>n_k-1$. This is only possible when the first two inequalities are equalities, i.e. $f(n_k)\mid n_k=kf(n_k)$. The numbers n_k are different, which finishes the proof.

3. Let H,I and G be the feet of the perpendiculars from Q to CB,CA and from P to AB, respectively. Since the quadrilaterals $AEPG$ and $AFQI$ are similar, we have $\angle AEG=\angle AFI$, therefore E,F,G,I lie on some circle k. Analogously, points D, E,I,H lie on a circle k_1, and points D, H,F,G on a circle k_2. If the circles k,k_1 and k_2 are distinct, their radical axes are AB,BC and CA, but they are neither congruent nor parallel, a contradiction. Therefore $k_1\equiv k_2\equiv k$ and D,E,F,G, H,I all lie on a single circle.



Let K and L be the centers of circles $CDPE$ and PFG. Since $MD\cdot ME=MF\cdot MG$, line MP is the radical axis of these two circles, so it is perpendicular to KL. Since K and L are the midpoints of PC and PF, the lines KL and CF are parallel, hence $MP\perp CF$.

Mathematical Competitions in Serbia

http://srb.imomath.com/

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