**THE 4th ROMANIAN MASTER OF MATHEMATICS COMPETITION**

DAY 1: FRIDAY, FEBRUARY 25, 2011, BUCHAREST

Language: English

**Problem 1.** Prove that there exist two functions $f, g\colon \mathbb{R} \to \mathbb{R}$, such that $f \circ g$ is strictly decreasing and $g \circ f$ is strictly increasing.

**Problem 2.** Determine all positive integers $n$ for which there exists a polynomial $f(x)$ with real coefficients, with the following properties:

(1) for each integer $k$, the number $f(k)$ is an integer if and only if $k$ is not divisible by $n$;
(2) the degree of $f$ is less than $n$.

**Problem 3.** A triangle $ABC$ is inscribed in a circle $\omega$. A variable line $\ell$ chosen parallel to $BC$ meets segments $AB$, $AC$ at points $D$, $E$ respectively, and meets $\omega$ at points $K$, $L$ (where $D$ lies between $K$ and $E$). Circle $\gamma_1$ is tangent to the segments $KD$ and $BD$ and also tangent to $\omega$, while circle $\gamma_2$ is tangent to the segments $LE$ and $CE$ and also tangent to $\omega$. Determine the locus, as $\ell$ varies, of the meeting point of the common inner tangents to $\gamma_1$ and $\gamma_2$.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.

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DAY 2: SATURDAY, FEBRUARY 26, 2011, BUCHAREST

Language: English

**Problem 4.** Given a positive integer $n = \prod_{i=1}^{s} p_i^{\alpha_i}$, we write $\Omega(n)$ for the total number $\sum_{i=1}^{s} \alpha_i$ of prime factors of $n$, counted with multiplicity. Let $\lambda(n) = (-1)^{\Omega(n)}$ (so, for example, $\lambda(12) = \lambda(2^2 \cdot 3^1) = (-1)^{2+1} = -1$).

Prove the following two claims:

i) There are infinitely many positive integers $n$ such that $\lambda(n) = \lambda(n+1) = +1$;
ii) There are infinitely many positive integers $n$ such that $\lambda(n) = \lambda(n+1) = -1$.

**Problem 5.** For every $n \geq 3$, determine all the configurations of $n$ distinct points $X_1, X_2, \ldots, X_n$ in the plane, with the property that for any pair of distinct points $X_i$, $X_j$ there exists a permutation $\sigma$ of the integers $\{1, \ldots, n\}$, such that $\mathrm{d}(X_i, X_k) = \mathrm{d}(X_j, X_{\sigma(k)})$ for all $1 \leq k \leq n$.
(We write $\mathrm{d}(X, Y)$ to denote the distance between points $X$ and $Y$.)

**Problem 6.** The cells of a square $2011 \times 2011$ array are labelled with the integers $1, 2, \ldots, 2011^2$, in such a way that every label is used exactly once. We then identify the left-hand and right-hand edges, and then the top and bottom, in the normal way to form a torus (the surface of a doughnut).

Determine the largest positive integer $M$ such that, no matter which labelling we choose, there exist two neighbouring cells with the difference of their labels at least $M$.*

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.

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*Cells with coordinates $(x, y)$ and $(x', y')$ are considered to be neighbours if $x = x'$ and $y - y' \equiv \pm 1 \pmod{2011}$, or if $y = y'$ and $x - x' \equiv \pm 1 \pmod{2011}$.