Serbian Mathematical Olympiad 2010

for high school students

Niš, April 6–7, 2010

Problems and Solutions

Cover photo: Niš Fortress

SHORT HISTORY AND SYSTEM

Mathematical competitions in Serbia have been held since 1958. In the first years only republic competitions within the former Yugoslavia, which Serbia was a part of, were held. The first Federal Mathematical Competition in Yugoslavia was held in Belgrade in 1960, and since then it was held regularly every year, skipping only 1999 for a non-mathematical reason. The system has undergone relatively few changes. The earliest Federal Competitions were organized for 3rd and 4th grades of high school only; 2nd grade was added in 1970, and 1st grade in 1974. Since 1982, 3rd and 4th grades compose a single category. After the breakdown of the old Yugoslavia in 1991, the entire system was continued in the newly formed FR Yugoslavia, later renamed Serbia and Montenegro. The separation of Montenegro finally made the federal competition senseless as such. Thus, starting with 2007, the federal competition and team selection exam are replaced by a two-day Serbian Mathematical Olympiad.

Today a mathematical competition season in Serbia consists of four rounds of increasing difficulty:

- Municipal round, held in early February. The contest consists of 5 problems, each 20 points worth, to be solved in 3 hours. Students who perform well qualify for the next round (50-60 points are usually enough).
- Regional round, held in late February in the same format as the municipal round. Each student's score is added to that from the municipal round. Although the number of students qualified for the state round is bounded by regional quotas, a total score of 110-120 should suffice.
- State (republic) round, held in late March in a selected town in the country. There are roughly 200 to 300 participants. The contest consists of 5 problems in 4 hours.
- Serbian Mathematical Olympiad (SMO), held in early April in a selected place in the country. The participants are selected through the state round: 26 from A category (distribution among grades: 3+5+8+10), 3 from B category (0+0+1+2), plus those members of the last year's olympic team who did not manage to qualify otherwise. Six most successful contestants are invited to the olympic team.

Since 1998, contests for each grade on the three preliminary rounds are divided into categories A (specialized schools and classes) and B (others). A student from category B is normally allowed to work the problems for category A instead. On the SMO, all participants work on the same problems.

The Serbian Mathematical Olympiad 2010 for high school students took place in Niš on April 6–7. There were 29 students from Serbia and 9 guest students from Republic of Srpska (Bosnia and Herzegovina), the specialized school "Kolmogorov" in Moscow, and the host city of Niš taking part on the competition. The average score on the contest was 12.87 points and all problems were fully solved by the contestants, except problem 6 on which the maximum achieved score was 2. Based on the results of the competition the team of Serbia for the 27-th Balkan Mathematical Olympiad and the 51-st International Mathematical Olympiad was selected:

Teodor von Burg	Math High School, Belgrade	36 points
Luka Milićević	Math High School, Belgrade	35 points
Rade Špegar	Math High School, Belgrade	32 points
Mihajlo Cekić	Math High School, Belgrade	30 points
Stevan Gajović	Math High School, Belgrade	29 points
Dušan Milijančević	Math High School, Belgrade	26 points

In this booklet we present the problems and full solutions of the Serbian Mathematical Olympiad and the Balkan Mathematical Olympiad.

SERBIAN MATHEMATICAL OLYMPIAD

for high school students

Niš , 06.04.2010.

First Day

1. Some of n towns are connected by two-way airlines. There are m airlines in total. For $i = 1, 2, \ldots, n$, let d_i be the number of airlines going from town i. If $1 \le d_i \le 2010$ for each $i = 1, 2, \ldots, 2010$, prove that

$$\sum_{i=1}^{n} d_i^2 \le 4022m - 2010n.$$

Find all n for which equality can be attained. *(Aleksandar Ilić)*

2. In an acute-angled triangle ABC, M is the midpoint of side BC, and D, E and F the feet of the altitudes from A, B and C, respectively. Let H be the orthocenter of $\triangle ABC$, S the midpoint of AH, and G the intersection of FE and AH. If N is the intersection of the median AM and the circumcircle of $\triangle BCH$, prove that $\measuredangle HMA = \measuredangle GNS$. *(Marko Djikić)*

3. Let A be an infinite set of positive integers. Find all natural numbers n such that for each $a \in A$

$$a^n + a^{n-1} + \cdots + a^1 + 1 \mid a^{n!} + a^{(n-1)!} + \cdots + a^{1!} + 1.$$

(Miloš Milosavljević)

Time allowed: 270 minutes.
Each problem is worth 7 points.

SERBIAN MATHEMATICAL OLYMPIAD

for high school students

Niš, 07.04.2010.

Second Day

4. Let O be the circumcenter of triangle ABC. A line through O intersects the sides CA and CB at points D and E respectively, and meets the circumcircle again at point $P \neq O$ inside the triangle. A point Q on side AB is such that $\dfrac{AQ}{QB} = \dfrac{DP}{PE}$. Prove that $\measuredangle APQ = 2\measuredangle CAP$. *(Dušan Djukić)*

5. An $n \times n$ table whose cells are numerated with numbers $1, 2, \ldots, n^2$ in some order is called *Naissus* if all products of n numbers written in n *scattered* cells give the same residue when divided by $n^2 + 1$. Does there exist a Naissus table for

(a) $n = 8$;

(b) $n = 10$?

(n cells are *scattered* if no two are in the same row or column.) *(Marko Djikić)*

6. Let a_0 and a_n be different divisors of a natural number m, and $a_0, a_1, a_2, \ldots, a_n$ be a sequence of natural numbers such that it satisfies

$$a_{i+1} = |a_i \pm a_{i-1}| \quad \text{for } 0 < i < n.$$

If $\gcd(a_0, \ldots, a_n) = 1$, show that there is a term of the sequence that is smaller than $\sqrt{m}$. *(Dušan Djukić)*

Time allowed: 270 minutes.
Each problem is worth 7 points.

SOLUTIONS

1. By the problem conditions, $0 \leq (d_i - 1)(2010 - d_i)$ holds for each i, i.e. $d_i^2 \leq 2011d_i - 2010$. Since $\sum_{i=1}^n d_i = 2m$, summing up these inequalities gives

$$\sum_{i=1}^{n} d_i^2 \leq 2011 \cdot \sum_{i=1}^{n} d_i - 2010n = 4022m - 2010n,$$

as desired.

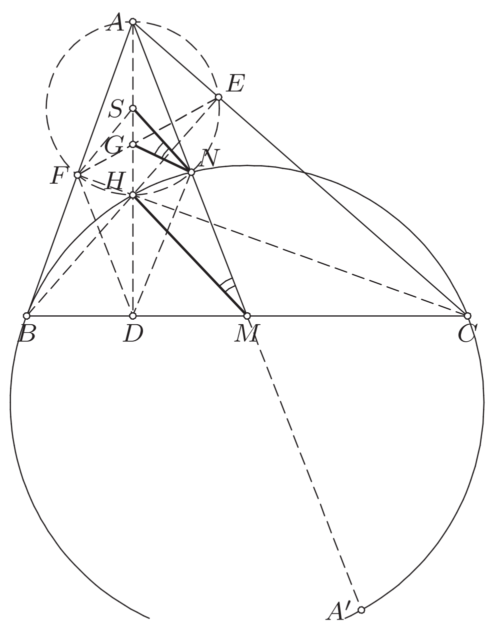
The equality holds if and only if $d_i \in \{1, 2010\}$ for each $i \in \{1, 2, \dots, n\}$.

1° Let $n = 2k$ for some $k \in \mathbb{N}$. Setting an airline between towns i and $i + k$ for $i = 1, \dots, k$ and no other airlines yields a configuration with $d_i = 1$ for all i.

2° Let $n = 2k - 1$ for some $k \in \mathbb{N}$. We cannot have $d_i = 1$ for all i because the sum of all d_i must be even, so we must have $d_j = 2010$ for some j; hence $n \geqslant 2011$. On the other hand, setting an airline between towns $2i$ and $2i+1$ for $i = 1006, \dots, k$ and between 1 and i for $1 \leq i \leq 2010$ yields a configuration with $d_1 = 2010$ and $d_i = 1$ for $i = 2, \dots, n$.

Therefore the equality can be attained if and only if $2 \mid n$, or $2 \nmid n$ and $n \geq 2011$.

2. Let the line AN meet the circle BHC again at point A'. Then $ABA'C$ is a parallelogram and $\angle HCA' = \angle HCB + \angle BCA' = \angle HCB + \angle ABC = 90^\circ$, so the points A, B, N all lie on the circle with diameter HA' and therefore $\angle ANH = 90^\circ$.



Since S is the circumcenter of $\triangle AEF$, it follows that $\angle SFG = 90^\circ - \angle EAF = \angle ACF = \angle ADF$; hence the triangles SFG and SDF are similar and $SG \cdot SD = SF^2 = SN^2$. This in turn shows that $\triangle SNG \sim \triangle SDN$ and finally $\angle GNS = \angle GDN = \angle HMN$ since the quadrilateral $HDMN$ is cyclic.

Second solution. Quadrilaterals $BDHF$ and $DCEH$ are cyclic and $AF \cdot AB = AH \cdot AD = AE \cdot AC$. We apply the inversion $\mathcal{I}$ with center A and power $AF \cdot AB$. Clearly $\mathcal{I}(F) = B, \mathcal{I}(H) = D, \mathcal{I}(E) = C$, so $\mathcal{I}$ maps line BC to the circumcircle of

$\triangle FHE$, i.e. the circle with diameter AH; also, $\mathcal{I}$ maps the circumcircle of BCH to the circumcircle ω of FDE which is the nine-point circle of $\triangle ABC$. Since $M \in \omega \cap AM$ and $\mathcal{I}$ preserves line AM, it follows that $\mathcal{I}(M) = N$.

Let $\mathcal{I}(G) = G^*$ and $\mathcal{I}(S) = S^*$. Since $\mathcal{I}(EF)$ is the circumcircle of $\triangle ABC$, $S \in \omega$ and $\mathcal{I}(AH) = AH$, points G^* and S^* are the second intersection points of AH with the circumcircles of $\triangle ABC$ and $\triangle HBC$ respectively. Thus $\angle GNS = \angle G^*MS^* = \angle HMA$, for G^*, S^* are the mirrors of H, A in BC.

3. Consider polynomials $P(x) = x^n + x^{n-1} + \cdots + 1$ and $Q(x) = x^{n!} + \cdots + x^{1!} + 1$, and let $Q(x) = C(x)P(x) + R(x)$, where C and R are polynomials with integer coefficients and $\deg R < \deg P$. The problem condition claims that $P(a)$ divides $Q(a)$ and hence also divides $R(a)$ for infinitely many a, but for sufficiently large a we have $|R(a)| < |P(a)|$, implying $R(a) = 0$; thus $R(x)$ has infinitely many zeros, so $R(x) \equiv 0$ and $P(x) \mid Q(x)$.

> *Lemma.* Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{N}$. Polynomial $P(x)$ divides $Q(x) = x^{\alpha_n} + x^{\alpha_{n-1}} + \cdots + 1$ if and only if $\{0, \alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a complete residue system modulo $n+1$.
>
> *Proof.* Let r_i be the residue of α_i modulo $n+1$. Suppose that $P(x) \mid Q(x)$. Since $P(x)$ divides $x^{n+1} - 1$ which divides $x^{\alpha_i} - x^{r_i}$, it also divides $Q_1(x) = x^{r_1} + \cdots + x^{r_n} + 1$. Since $\deg Q_1 \leq \deg P$, it follows that $Q_1 = c \cdot P$ for some constant c, and this is obviously only possible if $c = 1$ and $\{r_1, \ldots, r_n\} = \{1, \ldots, n\}$. The other direction is trivial.

We conclude from the lemma that the desired numbers n are those for which $\{0, 1!, \ldots, n!\}$ is a complete residue system modulo $n+1$.

If $n > 3$ and $n+1$ is a composite number, then $n+1 \mid n!$, so this case is impossible. If $n+1 = p > 3$ is a prime, then by Wilson's theorem $(p-1)! \equiv -1 \pmod{p}$, so $(p-2)! \equiv 1 = 1! \pmod{p}$, again impossible. The only remaining possibilities are $n = 1$ and $n = 2$, and it is directly verified that these values satisfiy the conditions of the problem.

Second solution. We shall prove a stronger statement: If $A = a^n + \cdots + a + 1$ divides $a^{n!} + \cdots + a^{1!} + 1$ for some integer $a > 1$, then $n \in \{1, 2\}$.

Suppose that A divides $a^{\alpha_1} + \cdots + a^{\alpha_n} + 1$ for some positive integers $\alpha_1, \ldots, \alpha_n$, and let r_i be the remainder of α_i upon division by $n+1$. Like in the first solution, A divides $C = a^{r_1} + \cdots + a^{r_n} + 1 = c_n a^n + \cdots + c_1 a + c_0$, where $c_0, \ldots, c_n$ are nonnegative integers with the sum $n+1$.

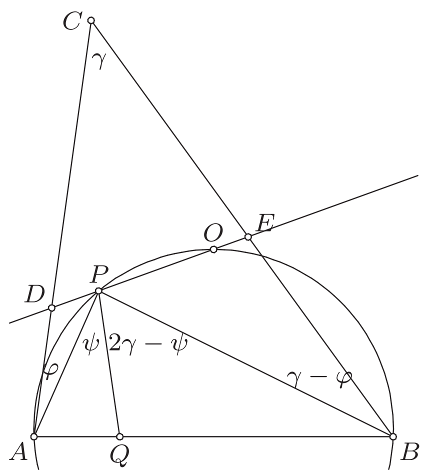
Of all numbers of the form $B = b_n a^n + \cdots + b_1 a + b_0$ that are divisible by A ($b_i \in \mathbb{N}_0$), consider any one with the minimum possible value of the sum $b_0 + \cdots + b_n$. Each b_i is less than a, otherwise substituting (b_i, b_{i+1}) with $(b_i - n, b_{i+1} + 1)$ yields a representation of B with a smaller sum $b_0 + \cdots + b_n$ (where $b_{n+1} = b_0$). Therefore $B < a \cdot A$, which implies $B = kA$ for some $k \in \{1, \ldots, n\}$, so $b_i = k$ by uniqueness of base a representation. It follows that the minimum possible value of $b_0 + \cdots + b_n$ equals $n + 1$ and is attained only when $b_i = 1$ for all i.

Since $c_0 + \cdots + c_n = n + 1$, it follows from above that $c_i = 1$ for all i. Thus $\{r_1, \ldots, r_n\} = \{1, \ldots, n\}$ and we can proceed like in the first solution to obtain $n \in \{1, 2\}$.

4. The given configuration is only possible if $\triangle ABC$ is acute-angled. Denote $\angle PAD = \varphi$, $\angle QPA = \psi$, $\angle BCA = \gamma$. Then $\angle APB = 2\gamma$ and $\angle DAP + \angle EBP = \angle APB - \angle ACB = \gamma$, so $\angle PBE = \gamma - \varphi$ and $\angle BPQ = 2\gamma - \psi$. Since $\angle APD = \angle BPE = 90^\circ - \gamma$, we also have $\angle ADP = 90^\circ + \gamma - \varphi$ and $\angle BEP = 90^\circ + \varphi$.

The sine theorem in triangles APD and PBE gives us

$$\frac{DP}{PE} = \frac{DP}{PA} \cdot \frac{PA}{PB} \cdot \frac{PB}{PE} = \frac{\sin\varphi\cos\varphi}{\sin(\gamma-\varphi)\cos(\gamma-\varphi)} \cdot \frac{PA}{PB} = \frac{\sin 2\varphi}{\sin(2\gamma - 2\varphi)} \cdot \frac{PA}{PB}.$$



On the other hand, $\dfrac{AQ}{QB} = \dfrac{AQ}{AP} \cdot \dfrac{AP}{BP} \cdot \dfrac{BP}{QB} = \dfrac{\sin\psi}{\sin(2\gamma - \psi)} \cdot \dfrac{AP}{PB}$.

Now $\dfrac{AQ}{QB} = \dfrac{DP}{PE}$ implies $f(2\varphi) = f(\psi)$, where $f(x) = \dfrac{\sin x}{\sin(2\gamma - x)}$. However, f is strictly increasing on $(0, 2\gamma)$ because $f'(x) = \dfrac{\sin(2\gamma - 2x)}{(\sin(2\gamma - x))^2} > 0$, so it immediately follows that $\psi = 2\varphi$.

5. Suppose that there exists a Naissus table 8×8 and that the product of any 8 scattered numbers gives a remainder r modulo $8^2 + 1 = 65 = 5 \cdot 13$. Since these numbers can be chosen so as to contain a multiple of 13, r is divisible by 13. On the other hand, the table can be partitioned into 8 pairwise disjoint groups of 8 scattered cells each, and at least one of these groups does not contain any multiple of 13, so the product over this group of scattered cells is not divisible by 13, a contradiction. Hence there is no Naissus table for $n = 8$.

For $n = 10$, $n^2 + 1 = 101$ is a prime, and there is a primitive root g modulo 101. Then the table $(a_{i,j})_{0 \le i,j \le 9}$ such that $a_{i,j} \equiv g^{10i+j} \pmod{101}$ contains each number from $\{1, \ldots, 100\}$ exactly once. Now let $a_{0,p(0)}, \ldots, a_{9,p(9)}$ be scattered numbers; note that $p(0), \ldots, p(9)$ is a permutation of $0, \ldots, 9$, so $p(0) + \cdots + p(9) = 45$.

The product of numbers $a_{i,p(i)}$ is congruent to $g^{p(0)}g^{10+p(1)}\cdots g^{90+p(9)} = g^{450+45} = g^{495}$ no matter which scattered cells are chosen, so this is an example of a Naissus table.

6. Let p and q be two smallest (different) terms of the sequence. Assume that $a_k = p$ and $a_l = q$ $(k < l)$ and that (without loss of generality) $a_m \notin \{p, q\}$ for $k < m < l$. If $l > k+2$ then $a_{k+3} = a_{k+2} + a_{k+1}$ (otherwise $a_{k+3} = |a_{k+2} - a_{k+1}| = p$) and similarly $a_{l-3} = a_{l-2} + a_{l-1}$. Now if a_m is the maximal term between a_k and a_l, we have $k+2 < m < l-2$, so $a_{m+2} = a_m - a_{m+1} = a_{m-1}$ and $a_{m+1} = a_m - a_{m-1} = a_{m-2}$. Thus the sequence obtained by excluding terms a_m, a_{m+1}, a_{m+2} will still satisfy the imposed conditions. Substituting 5-tuples $(u, v, u+v, u, v)$ in the sequence with pairs (u, v) can be repeated until there is at most one term remaining between p and q. However, if we go backward now, we will have to insert either p or q in between. This leads to the concludion that in the original sequence there could be at most one term between $a_k = p$ and $a_l = q$; hence either $l = k+1$ or $l = k+2$ (in the latter case, $a_{k+1} = p+q$).

Denote $v_k = (1, 0)$, $v_l = (0, 1)$ and, for each i, $v_{i+2} = \varepsilon_i v_i + \varepsilon_i' v_{i+1}$ if $a_{i+2} = \varepsilon_i a_i + \varepsilon_i' a_{i+1}$ $(\varepsilon_i, \varepsilon_i' \in \{-1, 1\})$. A simple induction shows that if $v_i = (x_i, y_i)$ then $a_i = x_i p + y_i a_{k+1}$. Since $x_{i+1}y_{i+2} - x_{i+2}y_{i+1} = x_{i+1}(\varepsilon_i y_i + \varepsilon_i' y_{i+1}) - (\varepsilon_i x_i + \varepsilon_i' x_{i+1})y_{i+1} = -\varepsilon_i(x_i y_{i+1} - x_{i+1}y_i)$, it follows that for each i we have $x_i y_{i+1} - x_{i+1} y_i \in \{-1, 1\}$, so $(x_i, y_i) = 1$. Suppose that $x_i < 0$ or $y_i < 0$ for some $i > k$ (at least one of x_i, y_i must be positive because so is a_i), and let j be the smallest such i. Since $x_j y_{j-1} - x_{j-1} y_j \in \{-1, 1\}$ (i $x_j y_j < 0$), it follows that $v_{j-1} \in \{(0,1), (1,0)\}$, so keeping in mind that either $v_j = v_{j-1} - v_{j-2}$ or $v_j = v_{j-2} - v_{j-1}$ we can deduce that $v_{j-2} \in \{(1,0), (0,1)\}$. (Indeed, $x_{j-2}, y_{j-2} \geq 0$, so if we assume e.g. $v_{j-1} = (0, 1)$, relation $x_{j-2}y_{j-1} - x_{j-1}y_{j-2} \in \{-1, 1\}$ will imply $x_{j-2} = 1$, which together with $v_j = \pm(v_{j-1} - v_{j-2})$ and $x_j y_j < 0$ implies $y_{j-2} = 0$.) It follows that $v_j \in \{(-1, 1), (1, -1)\}$ and $a_j = |p - q| < \max\{p, q\}$, contradicting the choice of p and q.

This shows that, if p, q are two smallest terms of the sequence, then each term is of the form $a_i = x_i p + y_i q$ with $x_i, y_i \in \mathbb{N}$ and $(x_i, y_i) = 1$. Note that p and q must be coprime. Now let $m = da_0 = ea_n$. There exist $x_0, y_0, x_n, y_n \in \mathbb{N}$ such that $(x_0, y_0) = (x_n, y_n) = 1$ and $a_0 = x_0 p + y_0 q$, $a_n = x_n p + y_n q$, so $m = dx_0 p + dy_0 q = ex_n p + ey_n q$ and $(dx_0, dy_0) \neq (ex_n, ey_n)$. Consequently, $p \mid dy_0 - ey_n$, so $dy_0 > p$ or $ey_n > p$, which finally implies $m > pq$ and therefore $\min(p, q) < \sqrt{m}$.

The 27-th Balkan Mathematical Olympiad was held from May 2 to May 8 in Chisinau in Moldova. The results of Serbian contestants are given in the following table:

	1	2	3	4	Total	
Teodor von Burg	10	10	10	8	38	Gold Medal
Luka Milićević	10	10	9	7	36	Gold Medal
Rade Špegar	10	1	10	2	23	Bronze Medal
Mihajlo Cekić	10	10	1	0	21	Bronze Medal
Stevan Gajović	10	10	4	2	26	Silver Medal
Dušan Milijančević	10	10	1	4	25	Silver Medal

After the contest, 8 contestants (7 officially + 1 unofficially) with 35-40 points were awarded gold medals, 24 (13+11) with 25-34 points were awarded silver medals, and 43 (19+24) with 12-24 points were awarded bronze medals. The most successful contestant was Teodor von Burg from Serbia with 38 points.

The unofficial ranking of the teams is given below:

Member Countries		Guest Teams	
1. Romania	186	Kazakhstan	160
2. Serbia	169	Italy	137
3. Bulgaria	159	United Kingdom	115
4. Turkey	151	France	112
5. Moldova	128	Azerbaijan	87
6. Greece	94	Turkmenistan	79
7. FYR Macedonia	79	Moldova 2	66
8. Cyprus	58	Saudi Arabia	66
9. Montenegro	16	Tajikistan	55
10. Albania	10		

BALKAN MATHEMATICAL OLYMPIAD

Chişinău, Moldova , 04.05.2010.

1. If a, b and c are positive real numbers, prove that

$$\frac{a^2b(b-c)}{a+b}+\frac{b^2c(c-a)}{b+c}+\frac{c^2a(a-b)}{c+a}\geq 0$$ *(Saudi Arabia)*

2. Let ABC be an acute-angled triangle with orthocenter H and let M be the midpoint of AC. The foot of the altitude from C is C_1. Point H_1 is symmetric to H in AB. The projections of C_1 on lines AH_1, AC and BC are P, Q and R respectively. If M_1 is the circumcenter of triangle PQR, prove that the point symmetric to M with respect to M_1 lies on line BH_1. *(Serbia)*

3. We define a *w-strip* as the set of all points in the plane that are between or on two parallel lines on a mutual distance w. Let S be a set of n points in the plane such that any three points from S can be covered by a 1-strip. Show that the entire set S can be covered by a 2-strip. *(Romania)*

4. For every integer $n \geq 2$, denote by $f(n)$ the sum of positive integers not exceeding n that are not coprime to n. Prove that $f(n+p) \neq f(n)$ for any such n and any prime number p. *(Turkey)*

Time allowed: 270 minutes.
Each problem is worth 10 points.

SOLUTIONS

1. The left-hand side is equal to

$$\frac{a^3b^3 + b^3c^3 + c^3a^3 - a^3b^2c - b^3c^2a - c^3a^2b}{(a+b)(b+c)(c+a)},$$

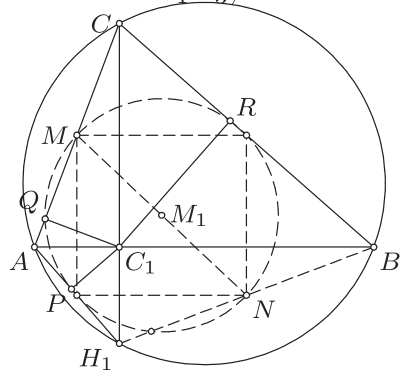
so it is enough to show that $a^3b^3+b^3c^3+c^3a^3 \ge a^3b^2c+b^3c^2a+c^3a^2b$. The AM-GM inequality gives us $a^3b^3 + a^3b^3 + a^3c^3 \ge 3\sqrt[3]{a^3b^3 \cdot a^3b^3 \cdot a^3c^3} = 3a^3b^2c$; summing this inequality and its cyclic analogs yields the desired inequality.
Equality holds if and only if $a = b = c$.

2. We shall use the following simple statement.

Lemma. Let $A_1A_2A_3A_4$ be a convex cyclic quadrilateral whose diagonals are orthogonal and meet at X. If B_i is the midpoint of side A_iA_{i+1} and X_i the projection of X on this side ($A_5 = A_1$), then the eight points B_i, X_i ($i = 1, 2, 3, 4$) lie on a circle.

Proof. Quadrilateral $B_1B_2B_3B_4$ is a rectangle because $B_1B_2 \parallel B_3B_4 \parallel A_1A_3$ and $B_2B_3 \parallel B_4B_1 \parallel A_2A_4$. Denote by k the circumcircle of $B_1B_2B_3B_4$. Since $\angle B_3XA_3 = \angle A_4A_3A_1 = \angle A_4A_2A_1 = \angle A_1XX_1$, points B_3, X, X_1 are collinear, so X_1 lies on the circle with diameter B_1B_3, i.e. k. Similarly, X_2, X_3, X_4 lie on k.

It is known that H_1 lies on the circumcircle of ABC. By the lemma, points P, Q, R all lie on the circle with diameter MN, where N is the midpoint of BH_1. Therefore N is symmetric to M with respect to M_1 and lies on BH_1 as desired.



3. Of all triangles with the vertices in S, consider one with a maximum area, say $\triangle ABC$. Let A', B', C' be the points symmetric to A, B, C with respect to the midpoints of BC, CA, AB, respectively. We claim that all points from S must lie inside or on the boundary of $\triangle A'B'C'$. Indeed, if $X \in S$ is outside $\triangle A'B'C'$, we can assume without loss of generality that X and BC are on different sides of $B'C'$, and then $\triangle BCX$ has an area greater than $\triangle ABC$, a contradiction.

The triangle ABC can be covered by a 1-strip, so the triangle $A'B'C'$, being similar to ABC with ratio 2, can be covered by a 2-strip, also covering all of S.

4. There are $n+1-\varphi(n)$ nonnegative integers not coprime with n, and whenever r is among them, so is $n-r$. This gives us the formula $f(n)=\frac{1}{2}n(n+1-\varphi(n))$. Suppose that $f(n)=f(n+p)$. We observe first that n and $n+p$ divide $2f(n)<n(n+p)$, so n and $n+p$ are not coprime, which implies that $n=kp$ for some $k\in\mathbb{N}$. Then the equality $f(n)=f(n+p)$ is equivalent to $k(kp+1-\varphi(kp))=(k+1)(kp+p+1-\varphi(kp+p))$, so

$$kp+1-\varphi(kp)=(k+1)x \quad \text{and} \quad kp+p+1-\varphi(kp+p)=kx \tag{1}$$

for some $x\in\mathbb{N}$, $x<p$. Subtraction gives us $x=\varphi(kp+p)-\varphi(kp)-p$. Since $\varphi(kp)$ and $\varphi(kp+p)$ are both divisible by $p-1$ (by the formula for $\varphi(n)$), we obtain $x\equiv -1 \pmod{p-1}$.

If $p=2$ then $x=1$ and $\varphi(2k+2)=k+3$, which is impossible because $\varphi(2k+2)\le k+1$. If $p=3$ then $x=1$ and $\varphi(3k+3)=2k+4$, again impossible because $\varphi(3k+3)\le 2k+2$. Therefore $p\ge 5$, so $x\equiv -1 \pmod{p-1}$ implies $x=p-2$. Plugging this value in (1) leads to

$$\varphi(kp)=2k+3-p \quad \text{and} \quad \varphi(kp+p)=2k+1+p.$$

If k is divisible by p, then $\varphi(kp)$ is also divisible by p, so $p\mid 2k+3$ and hence $p\mid 3$, a contradiction. Similarly, $p\nmid k+1$. It follows that $\varphi(kp)=(p-1)\varphi(k)$ and $\varphi(kp+p)=(p-1)\varphi(k+1)$ which together with (1) yields

$$\varphi(k)=\frac{2k+2}{p-1}-1 \quad \text{and} \quad \varphi(k+1)=\frac{2k+2}{p-1}+1.$$

From here we see that $\varphi(t)$ is not divisible by 4 either for $t=k$ or for $t=k+1$, which is only possible if $t=q^i$ or $t=2q^i$ for some odd prime q and $i\in\mathbb{N}$, or $t\in\{1,2,4\}$. The cases $t=1,2,4$ are easily ruled out, so either k or $k+1$ is of the form q^i or $2q^i$. For $t=k=q^i$, $\varphi(q^i)+1=q^{i-1}(q-1)+1$ divides $2q^i+2$ which is impossible because $q^i+1>q^{i-1}(q-1)+1>\frac{2}{3}(2q^i+2)$. The other three cases are similarly shown to be impossible.