

THE 3RD ROMANIAN MASTER OF MATHEMATICS COMPETITION

DAY 1: FRIDAY, FEBRUARY 26, 2010, BUCHAREST

Language: English

Problem 1. For a finite non-empty set of primes P , let $m(P)$ be the largest possible number of consecutive positive integers, each of which is divisible by at least one member of P .

(i) Show that $|P| \leq m(P)$, with equality if and only if $\min(P) > |P|$;

(ii) Show that $m(P) < (|P| + 1)(2^{|P|} - 1)$.

(The number $|P|$ is the size of the set P .)

Problem 2. For each positive integer n , find the largest real number C_n with the following property. Given any n real-valued functions $f_1(x), f_2(x), \dots, f_n(x)$ defined on the closed interval $0 \leq x \leq 1$, one can find numbers x_1, x_2, \dots, x_n , such that $0 \leq x_i \leq 1$, satisfying

$$\left| f_1(x_1) + f_2(x_2) + \dots + f_n(x_n) - x_1 x_2 \dots x_n \right| \geq C_n.$$

Problem 3. Let $A_1 A_2 A_3 A_4$ be a convex quadrilateral with no pair of parallel sides. For each $i = 1, 2, 3, 4$, define ω_i to be the circle touching the quadrilateral externally, and which is tangent to the lines $A_{i-1} A_i$, $A_i A_{i+1}$ and $A_{i+1} A_{i+2}$ (indices are considered modulo 4, so $A_0 = A_4$, $A_5 = A_1$ and $A_6 = A_2$). Let T_i be the point of tangency of ω_i with the side $A_i A_{i+1}$. Prove that the lines $A_1 A_2, A_3 A_4$ and $T_2 T_4$ are concurrent if and only if the lines $A_2 A_3, A_4 A_1$ and $T_1 T_3$ are concurrent.

Each of the three problems is worth 7 points.

Time allowed: $4\frac{1}{2}$ hours.

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DAY 2: SATURDAY, FEBRUARY 27, 2010, BUCHAREST

Language: English

Problem 4. Determine whether there exist a polynomial $f(x_1, x_2)$ in two variables, with integer coefficients, and two points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ in the plane, satisfying all the following conditions:

- (i) A is an integer point (i.e., a_1 and a_2 are integers);
- (ii) $|a_1 - b_1| + |a_2 - b_2| = 2010$;
- (iii) $f(n_1, n_2) > f(a_1, a_2)$, for all integer points (n_1, n_2) in the plane other than A ;
- (iv) $f(x_1, x_2) > f(b_1, b_2)$, for all points (x_1, x_2) in the plane other than B .

Problem 5. Let n be a given positive integer. Say that a set K of points with integer coordinates in the plane is *connected* if for every pair of points $R, S \in K$, there exist a positive integer ℓ and a sequence $R = T_0, T_1, \dots, T_\ell = S$ of points in K , where each T_i is distance 1 away from T_{i+1} . For such a set K , we define the set of vectors

$$\Delta(K) = \{\overrightarrow{RS} \mid R, S \in K\}.$$

What is the maximum value of $|\Delta(K)|$ over all connected sets K of $2n + 1$ points with integer coordinates in the plane?

Problem 6. Given a polynomial $f(x)$ with rational coefficients, of degree $d \geq 2$, we define the sequence of sets $f^0(\mathbb{Q}), f^1(\mathbb{Q}), \dots$ by $f^0(\mathbb{Q}) = \mathbb{Q}$ and $f^{n+1}(\mathbb{Q}) = f(f^n(\mathbb{Q}))$ for $n \geq 0$. (Given a set S , we write $f(S)$ for the set $\{f(x) \mid x \in S\}$.)

Let $f^\omega(\mathbb{Q}) = \bigcap_{n=0}^{\infty} f^n(\mathbb{Q})$ be the set of numbers that are in all of the sets $f^n(\mathbb{Q})$. Prove that $f^\omega(\mathbb{Q})$ is a finite set.

Each of the three problems is worth 7 points.

Time allowed: $4\frac{1}{2}$ hours.