SERBIAN MATHEMATICAL OLYMPIAD

for high school students

Novi Sad , 13.04.2009.

First Day

1. In a scalene triangle ABC, α and β respectively denote the interior angles at vertices A and B. The bisectors of these two angles meet the opposite sides of the triangle at points D and E, respectively. Prove that the acute angle between the lines DE and AB does not exceed $\frac{|\alpha-\beta|}{3}$. *(Dušan Djukić)*

2. Find the smallest natural number which is a multiple of 2009 and whose sum of (decimal) digits equals 2009. *(Miloš Milosavljević)*

3. Determine the largest positive integer n for which there exist pairwise different sets $S_1, S_2, \ldots, S_n$ with the following properties:

1° $|S_i \cup S_j| \leq 2004$ for any two indices $1 \leq i, j \leq n$, and

2° $S_i \cup S_j \cup S_k = \{1, 2, \ldots, 2008\}$ for any $1 \leq i < j < k \leq n$. *(Ivan Matić)*

Time allowed: 270 minutes.
Each problem is worth 7 points.

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Novi Sad , 14.04.2009.

Second Day

4. For any $n \in \mathbb{N}$, denote by A_n the set of permutations $(a_1, a_2, \ldots, a_n)$ of set $\{1, 2, \ldots, n\}$ satisfying

$$k \mid 2(a_1 + a_2 + \cdots + a_k), \qquad \text{for each } 1 \leq k \leq n.$$

Compute the number of elements of A_n. *(Vidan Govedarica)*

5. Let x, y, z be arbitrary positive numbers such that $xy + yz + zx = x + y + z$. Prove that

$$\frac{1}{x^2 + y + 1} + \frac{1}{y^2 + z + 1} + \frac{1}{z^2 + x + 1} \leq 1.$$

When does equality occur? *(Marko Radovanović)*

6. The incircle k of a scalene triangle ABC is centered at S and tangent to the sides BC, CA, AB in points P, Q, R, respectively. Lines QR and BC intersect at point M. A circle passing through B and C is tangent to k at point N. The circumcircle of triangle MNP meets the line AP at point L different from P. Prove that the points S, L and M are collinear. *(Djordje Baralić)*

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