Serbian Mathematical Olympiad 2009

for high school students

Novi Sad, April 13–14, 2009

Problems and Solutions

SHORT HISTORY AND SYSTEM

Mathematical competitions in Serbia have been held since 1958. In the first few years only republic competitions within the former Yugoslavia, which Serbia was a part of, were held. The first Federal Mathematical Competition in Yugoslavia was held in Belgrade in 1960, and since then it has been held regularly every year, skipping only 1999 for a non-mathematical reason we prefer not to talk about here. The system has suffered relatively few changes. The earliest Federal Competitions were organized for 3rd and 4th grades of high school only; 2nd grade was added in 1970, and 1st grade in 1974. Since 1982, 3rd and 4th grades compose a single category. After the breakdown of the old Yugoslavia in 1991, the entire system was continued in the newly formed FR Yugoslavia, later renamed Serbia and Montenegro. The separation of Montenegro finally made the federal competition senseless as such. Thus, starting with 2007, the federal competition and team selection exam are replaced by a two-day Serbian Mathematical Olympiad.

Today a mathematical competition season in Serbia consists of four rounds of increasing difficulty:

- Municipal round, held in early February. The contest consists of 5 problems, each 20 points worth, to be solved in 3 hours. Students who perform well qualify for the next round (50-60 points are usually enough).
- Regional round, held in late February in the same format as the municipal round. Each student's score is added to that from the municipal round. Although the number of students qualified for the state round is bounded by regional quotas, a total score of 110-120 should suffice.
- State (republic) round, held in late March in a selected town in the country. There are roughly 200 to 300 participants. The contest consists of 5 problems in 4 hours.
- Serbian Mathematical Olympiad (SMO), held in middle April in a selected town in the country. The participants are selected through the state round: 26 from A category (distribution among grades: 3+5+8+10), 3 from B category (0+0+1+2), plus those members of the last year's olympic team who did not manage to qualify otherwise. Six most successful contestants are invited to the olympic team.

Since 1998, the problems for each grade on the three preliminary rounds are divided into A category (specialized schools and classes) and B category (others). A student from B category is normally allowed to work the problems for A category. On the SMO, all participants do the same problems.

The Serbian Mathematical Olympiad 2009 for high school students was held in Novi Sad on April 13–14. There were 31 students from Serbia and 3 guest students from the Serb entity of Bosnia and Herzegovina taking part on the competition. The contest was easier than last year in the sense that all problems were fully solved, with problems 2, 5 and 6 being the more difficult ones. Based on the results of the competition the team of Serbia for the 26-th Balkan Mathematical Olympiad and the 50-th International Mathematical Olympiad was selected:

Teodor von Burg	Math High School, Belgrade	32 points
Luka Milićević	Math High School, Belgrade	22 points
Dušan Milijančević	Math High School, Belgrade	21 points
Mihajlo Cekić	Math High School, Belgrade	20 points
Vukašin Stojisavljević	Math High School, Belgrade	20 points
Stefan Stojanović	HS "Svetozar Marković", Niš	17 points

In this booklet we present the problems and full solutions of the Serbian Mathematical Olympiad.

SERBIAN MATHEMATICAL OLYMPIAD

for high school students

Novi Sad, 13.04.2009.

First Day

1. In a scalene triangle ABC, α and β respectively denote the interior angles at vertices A and B. The bisectors of these two angles meet the opposite sides of the triangle at points D and E, respectively. Prove that the acute angle between the lines DE and AB does not exceed $\frac{|\alpha-\beta|}{3}$. *(Dušan Djukić)*

2. Find the smallest natural number which is a multiple of 2009 and whose sum of (decimal) digits equals 2009. *(Miloš Milosavljević)*

3. Determine the largest positive integer n for which there exist pairwise different sets $S_1, S_2, \ldots, S_n$ with the following properties:

1° $|S_i \cup S_j| \leq 2004$ for any two indices $1 \leq i, j \leq n$, and

2° $S_i \cup S_j \cup S_k = \{1, 2, \ldots, 2008\}$ for any $1 \leq i < j < k \leq n$. *(Ivan Matić)*

Time allowed: 270 minutes.
Each problem is worth 7 points.

SERBIAN MATHEMATICAL OLYMPIAD

for high school students

Novi Sad, 14.04.2009.

Second Day

4. For any $n \in \mathbb{N}$, denote by A_n the set of permutations $(a_1, a_2, \ldots, a_n)$ of set $\{1, 2, \ldots, n\}$ satisfying

$$k \mid 2(a_1 + a_2 + \cdots + a_k), \qquad \text{for each } 1 \le k \le n.$$

Compute the number of elements of A_n. *(Vidan Govedarica)*

5. Let x, y, z be arbitrary positive numbers such that $xy + yz + zx = x + y + z$. Prove that

$$\frac{1}{x^2 + y + 1} + \frac{1}{y^2 + z + 1} + \frac{1}{z^2 + x + 1} \le 1.$$

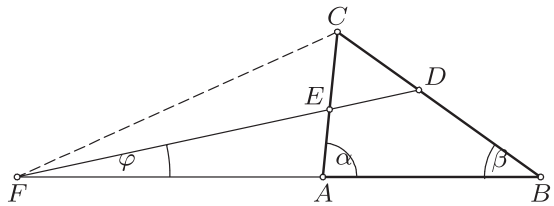
When does equality occur? *(Marko Radovanović)*

6. The incircle k of a scalene triangle ABC is centered at S and tangent to the sides BC, CA, AB in points P, Q, R, respectively. Lines QR and BC intersect at point M. A circle passing through B and C is tangent to k at point N. The circumcircle of triangle MNP meets the line AP at point L different from P. Prove that the points S, L and M are collinear. *(Djordje Baralić)*

Time allowed: 270 minutes.
Each problem is worth 7 points.

SOLUTIONS

1. Denote by γ the angle at C, and by a, b, c the corresponding sides of $\triangle ABC$. We assume without loss of generality that $\alpha > \beta$. Let F be the intersection of lines DE and AB and let φ be the angle between the two lines. By the known property of angle bisectors, we have $\frac{BD}{DC} = \frac{c}{b}$ and $\frac{CE}{EA} = \frac{a}{c}$, and consequently $BD = \frac{ac}{b+c}$, $DC = \frac{ab}{b+c}$, $CE = \frac{ab}{a+c}$ and $EA = \frac{bc}{a+c}$. Now the Menelaus theorem applied on line DE and triangle ABC gives us $AF = \frac{bc}{a-b}$ and $FB = \frac{ac}{a-b}$.



The sine theorem in triangles FEA and FDB gives us

$$\frac{\sin(\alpha-\varphi)}{\sin\varphi} = \frac{\sin\sphericalangle FEA}{\sin\sphericalangle EFA} = \frac{FA}{EA} = \frac{\frac{bc}{a-b}}{\frac{bc}{a+c}} = \frac{a+c}{a-b} \qquad \text{and}$$

$$\frac{\sin(\beta+\varphi)}{\sin\varphi} = \frac{\sin\sphericalangle FDB}{\sin\sphericalangle DFB} = \frac{FB}{DB} = \frac{\frac{ac}{a-b}}{\frac{ac}{b+c}} = \frac{b+c}{a-b},$$

from which we obtain $\sin\varphi = \sin(\alpha-\varphi) - \sin(\beta+\varphi) = 2\sin\frac{\alpha-\beta-2\varphi}{2}\cos\frac{\alpha+\beta}{2} < \sin(\alpha-\beta-2\varphi)$ and therefore $\varphi < \alpha-\beta-2\varphi$, which is equivalent to the desired inequality.

2. Since $2009 = 223 \cdot 9 + 2$, the required number has at least 224 digits. We shall consider the possible 224-digit numbers $x = \overline{c_{223}c_{222}\dots c_1c_0}$. Clearly $c_{223} \geq 2$. Also, if $c_{223} = 2$ then $c_{222} = \cdots = c_0 = 9$, so $x = 3 \cdot 10^{223} - 1 \equiv 3 \cdot 10 - 1 \equiv 1 \pmod 7$ is not divisible by $2009 = 7^2 \cdot 41$.

Suppose that $c_{223} = 3$. Then exactly one digit c_i, $i = 0, \dots, 222$ is equal to 8, while the others are equal to 9, so

$$x = 3\underbrace{99\dots9}_{222-i}8\underbrace{99\dots9}_{i} = 4 \cdot 10^{223} - 10^i - 1.$$

The order of 10 modulo 41 divides 40, and an easy inspection shows that $10^5 \equiv 1 \pmod{41}$, so 10^i is always congruent to one of $1, 10^1, 10^2, 10^3, 10^4$, i.e. to $1, 10, 18, 16, 37 \pmod{41}$. Therefore $x \equiv 4 \cdot 10^3 - 10^i - 1 \equiv 22 - 10^i \pmod{41}$ is never divisible by 41.

Now suppose that $c_{223} = 4$. Among $c_{222}, \dots, c_0$, either two are equal to 8, or one is equal to 7, while the others are 9. Thus $x = 5 \cdot 10^{223} - 10^i - 10^j - 1 \equiv 38 - (10^i + 10^j) \pmod{41}$, where i and j may be equal. It is easy to see from above that $10^i + 10^j \equiv$

38 (mod 41) if and only if one of $(i,j) \equiv (0,4)$ or $(4,0) \pmod 5$. In particular, $i \neq j$ and $i,j \leq 220$.

Next we assume that $j = 220$ and $i \equiv 4 \pmod 5$. Now we only need to choose i, if possible, so that $7^2 \mid x$. By Euler's theorem, $1 \equiv 10^{\varphi(49)} = 10^{42} \pmod{49}$, and since $10^k \equiv 10, 2, 20, 8, 31, 30, 48$ for $k = 1, 2, 3, 6, 7, 14, 21$ respectively, we deduce that the order of 10 modulo 49 equals 42. We have $x = 5 \cdot 10^{223} - 10^{220} - 10^i - 1 \equiv 5 \cdot 10^{13} - 10^{10} - 10^i - 1 \equiv 31 - 10^i \pmod{49}$, so $49 \mid x$ if and only if $10^i \equiv 31 \equiv 10^7 \pmod{49}$, i.e. $i \equiv 7 \pmod{42}$. The last condition together with $i \equiv 4 \pmod 5$ yields $i \equiv 49 \pmod{210}$, so the only possibility is $i = 49$. It follows that the required number is

$$4998\underbrace{9\ldots99}_{170}8\underbrace{99\ldots9}_{49}.$$

3. Each S_i has at most 2003 elements. Indeed, if some $|S_i| = 2004$, then $S_j \subset S_i$ for all j, which is impossible. Consider

$$G_{\{i,j\}} = \{1, 2, \ldots, 2008\} \setminus (S_i \cup S_j) \qquad \text{for } 1 \leq i, j \leq n.$$

Then $|G_{\{i,j\}}| \geqslant 4$ and $G_{\{i,j\}} \subseteq S_k$ for any pairwise distinct indices i, j, k, and $G_{\{i,j\}} \cap G_{\{k,l\}} \subseteq S_k \cap G_{\{k,l\}} = \emptyset$ whenever $i \neq j$, $k \neq l$ and $\{i,j\} \neq \{k,l\}$. It follows that

$$2008 \geq \left| \bigcup_{1 \leq i < j \leq n} G_{\{i,j\}} \right| = \sum_{1 \leq i < j \leq n} |G_{\{i,j\}}| \geq 4\binom{n}{2};$$

hence $\binom{n}{2} \leq 501$, so $n \leq 32$.

Now we shall construct 32 sets satisfying 1° and 2°.

Partition the set $\{1, 2, \ldots, 1984\}$ arbitrarily into 496 (disjoint) 4-element subsets; denote these subjects $G_{\{i,j\}}$, $1 \leq i < j \leq 496$ in an arbitrary manner. Define

$$S_i = \{1, 2, \ldots, 2008\} \setminus \bigcup_{j \neq i} G_{\{i,j\}} \qquad \text{for each } i = 1, \ldots, 32.$$

Condition 1° is automatically satisfied. Moreover, each $s \in \{1, 2, \ldots, 2008\}$ belongs to at most one $G_{\{u,v\}}$ which is contained in S_r for all $r \neq u, v$ (including at least one of i, j, k for any distinct i, j, k), so $s \in S_i \cup S_j \cup S_k$ for any $i < j < k$, which makes condition 2° satisfied as well.

Therefore the largest n satisfying the given conditions is $n = 32$.

4. Denote by F_n the number of elements of A_n. Then $F_1 = 1$, $F_2 = 2$, $F_3 = 6$.

Consider any permutation $(a_1, a_2, \ldots, a_n)$ from A_n, $n > 3$. Then $n-1$ must divide $2(a_1 + \cdots + a_{n-1}) = n(n+1) - 2a_n \equiv 2 - 2a_n \pmod{n-1}$, which implies that a_n is equal to 1, $\frac{n+1}{2}$ or n.

Suppose that $a_n = \frac{n-1}{2}$. Now $n-2$ divides $2(a_1 + \cdots + a_{n-2}) = n^2 - 1 - 2a_{n-1} \equiv 3 - 2a_{n-1} \pmod{n-2}$, so $2a_{n-1} - 3$ is divisible by $n-2$ and hence must be equal to $n-2$, but then $a_{n-1} = \frac{n+1}{2} = a_n$, which is impossible.

If $a_n = n$ then $(a_1, \ldots, a_n) \to (a_1, \ldots, a_{n-1})$ is a one-to-one correspondence with the elements of A_{n-1}, so there are F_{n-1} such permutations.

If $a_n = 1$ then $(a_1 - 1, \ldots, a_{n-1} - 1)$ is a permutation of $\{1, \ldots, n-1\}$. It belongs to A_{n-1} because $2\left((a_1 - 1) + \cdots + (a_k - 1)\right) = 2(a_1 + \cdots + a_k) - 2k$ is divisible by k for $k = 1, \ldots, n-1$. Again, this gives a one-to-one correspondence with the elements of A_{n-1}, so in this case there are also F_{n-1} permutations.

This establishes the recurrence $F_n = 2F_{n-1}$ for $n > 3$, which together with $F_3 = 6$ yields the formula $F_n = 3 \cdot 2^{n-2}$ for $n \geq 3$.

5. The Cauchy-Schwartz inequality for the triples $\left(x, \sqrt{y}, 1\right)$ and $\left(1, \sqrt{y}, z\right)$ gives us $\frac{1}{x^2+y+1} \leq \frac{1+y+z^2}{(x+y+z)^2}$. Analogously, $\frac{1}{y^2+z+1} \leq \frac{1+z+x^2}{(x+y+z)^2}$ and $\frac{1}{z^2+x+1} \leq \frac{1+x+y^2}{(x+y+z)^2}$. Summing up these three inequalities yields

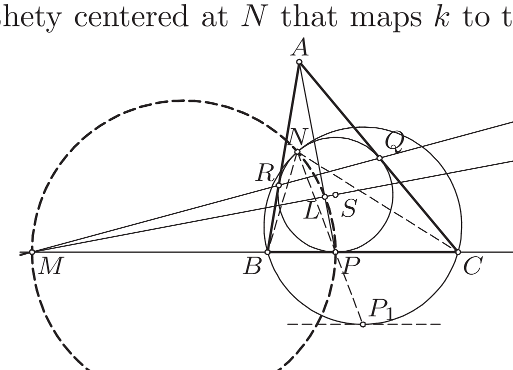
$$\frac{1}{x^2+y+1} + \frac{1}{y^2+z+1} + \frac{1}{z^2+x+1} \leq \frac{3+x+y+z+x^2+y^2+z^2}{(x+y+z)^2} = S.$$

All that remains is to show that $S \leq 1$, which is equivalent to

$$3 + x + y + z \leqslant 2(xy + yz + zx).$$

Since $xy + yz + zx = x + y + z$, the last inequality reduces to $x + y + z \geq 3$, which follows from $x + y + z = xy + yz + zx \leq \frac{(x+y+z)^2}{3}$.
Equality holds only for $x = y = z = 1$.

6. Let P_1 be the image of P under the homothety centered at N that maps k to the circle BCN. The tangent to circle BCN at P_1 is parallel to the tangent to k at P, i.e. the line BC, which means that P_1 is the midpoint of the arc BC of circle BCN. Thus NP bisects $\angle CNB$, so $\frac{BN}{CN} = \frac{BP}{CP}$. Since by the Menelaus theorem we have $\frac{BM}{MC} = \frac{BR}{RA} \cdot \frac{AQ}{QC} = \frac{BP}{PC} = \frac{BN}{NC}$, line NM is the external bisector of $\angle CNB$, and consequently N lies on the circle with diameter MP.



Denote by K the projection of S on AP. Points A, K, Q, R, S lie on the circle with diameter AS. The inversion Ψ with respect to k maps this circle to line QR, and also maps the circumcircle of $\triangle SKP$ to line BC, so Ψ maps K to point M. Therefore K lies on the line SM and $\angle MKP = 90°$. This means that the points K and L coincide, implying the statement of the problem.

Second solution. Consider the configuration of the problem in the complex plane, where S is at the origin and k is the unit circle. For any point X, we denote the corresponding complex number by x. Thus $s = 0$ and $|p| = |q| = |r| = 1$.

It is well-known that a point z lies on the line UV with $|u| = |v| = 1$ if and only if $z + uv\overline{z} = u + v$, and lies on the tangent to k at W with $|w| = 1$ na jedinichnu kruzhnicu ako i samo ako je $z + w^2\overline{z} = 2w$. From these facts we easily compute $m = \frac{p(pr+pq-2qr)}{p^2-rq}$ and $a = \frac{2qr}{q+r}$, $b = \frac{2pr}{p+r}$, $c = \frac{2pq}{p+q}$.

If O is the center of the circle BCN, we have $|n| = 1$, $\overline{o} = o/n^2$ and $|o - b| = |o - c| = |o - n|$. Relations $|o - b|^2 = |o - n|^2$ and $|o - c|^2 = |o - n|^2$ are reduced to $n^2(|b|^2 - 1) = o(b + n^2\overline{b} - 2n)$ and $n^2(|c|^2 - 1) = o(c + n^2\overline{c} - 2n)$, so $(|b|^2 - 1)(c + n^2\overline{c} - 2n) = (|c|^2 - 1)(b + n^2\overline{b} - 2n)$. Since $|b|^2 - 1 = -(\frac{p-r}{p+r})^2$ and $b + n^2\overline{b} - 2n = \frac{2(n-p)(n-r)}{p+r}$ and analogous relations hold for c, we obtain $(\frac{p-r}{p+r})^2\frac{2}{p+q}(n - q) = (\frac{p-q}{p+q})^2\frac{2}{p+r}(n - r)$ (since $n \neq p$). The solution of this linear equation in n upon simplification becomes

$$n = \frac{m+p}{p(\overline{m} + \overline{p})}.$$

The intersection X of AP and MS satisfies $\frac{x}{\overline{x}} = \frac{m}{\overline{m}}$ and $\frac{k-p}{\overline{k}-\overline{p}} = \frac{a-p}{\overline{a}-\overline{p}} = \frac{\frac{2rq}{r+q} - p}{\frac{2}{r+q} - \frac{1}{p}} = -\frac{m}{\overline{m}}$ so $SM \perp AP$, i.e. $PX \perp MX$. It remains to prove that $L \equiv X$, i.e. that X lies on the circumcircle of $\triangle MPN$. From above, this circle must be the circle with diameter MP, so it suffices to show that N is equidistant from M and P, which is shown directly from the above formula for n.

The 26-th Balkan Mathematical Olympiad was held from April 28 to May 4 in Kragujevac in Serbia. The results of Serbian contestants are given in the following table:

	1	2	3	4	Total	
Teodor von Burg	10	10	9	6	35	Gold Medal
Luka Milićević	10	10	1	8	29	Gold Medal
Dušan Milijančević	10	10	2	2	24	Gold Medal
Vukašin Stojisavljević	10	0	4	0	14	Bronze Medal
Mihajlo Cekić	10	10	0	0	20	Silver Medal
Stefan Stojanović	10	0	1	0	11	Bronze Medal

The host country usually sends another 6 contestants as a team B which takes part unofficially. The results of Serbian team B were as follows:

	1	2	3	4	Total	
Aleksandar Vasiljković	8	4	3	3	18	Silver Medal
Igor Spasojević	6	3	0	0	9	Bronze Medal
Filip Živanović	10	5	1	0	16	Silver Medal
Radomir Djoković	5	0	1	0	6	Bronze Medal
Stevan Gajović	10	4	3	1	18	Silver Medal
Vukan Levajac	2	0	0	0	2	

After the contest, 13 contestants (9 officially + 4 unofficially) with 24-40 points were awarded gold medals, 33 (15+18) with 15-22 points were awarded silver medals, and 42 (21+21) with 6-14 points were awarded bronze medals. The most successful contestant was Teodor von Burg from Serbia with 35 points.

The unofficial ranking of the teams is given below:

Member Countries		Guest Teams	
1. Serbia	133		
2. Turkey	131	Kazakhstan	112
3. Bulgaria	128	Italy	107
4. Romania	96	Azerbaijan	82
5. Moldova	76	Turkmenistan (5)	80
6. Greece	69	France	74
7. FYR Macedonia	59	Serbia B	69
8. Albania	36	United Kingdom	59
9. Bosnia and Herzegovina	33	Brno, Czech Rep.	42
10. Cyprus	22	Tajikistan (2)	30
11. Montenegro	9		

BALKAN MATHEMATICAL OLYMPIAD

Kragujevac , 30.04.2009.

1. Find all integer solutions of the equation

$$3^x - 5^y = z^2.$$

(Greece)

2. In a triangle ABC, points M and N on the sides AB and AC respectively are such that $MN \parallel BC$. Let BN and CM intersect at point P. The circumcircles of triangles BMP and CNP intersect at two distinct points P and Q. Prove that $\angle BAQ = \angle CAP$. *(Moldova)*

3. A 9×12 rectangle is divided into unit squares. The centers of all the unit squares, except the four corner squares and the eight squares adjacent (by side) to them, are colored red. Is it possible to numerate the red centers by $C_1, C_2, \ldots, C_{96}$ so that the following two conditions are fulfilled:

1° All segments $C_1C_2, C_2C_3, \ldots C_{95}C_{96}, C_{96}C_1$ have the length $\sqrt{13}$;

2° The poligonal line $C_1C_2 \ldots C_{96}C_1$ is centrally symmetric? *(Bulgaria)*

4. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$f\left(f(m)^2 + 2f(n)^2\right) = m^2 + 2n^2 \qquad \text{for all } m, n \in \mathbb{N}.$$

(Bulgaria)

Time allowed: 270 minutes.
Each problem is worth 10 points.

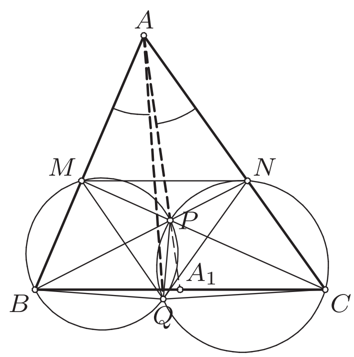
SOLUTIONS

1. We start by observing that z must be even, so $z^2 = 3^x - 5^y \equiv (-1)^x - 1 \pmod 4$ is divisible by 4, which implies that x is even, say $x = 2t$. Then our equation can be rewritten as $(3^t - z)(3^t + z) = 5^y$, which means that both $3^t - z = 5^k$ and $3^t + z = 5^{y-k}$ for some nonnegative integer k. Since $5^k + 5^{y-k} = 2 \cdot 3^t$ is not divisible by 5, it follows that $k = 0$ and

$$2 \cdot 3^t = 5^y + 1.$$

Suppose that $t \geq 2$. Then $5^y + 1$ is divisible by 9, which is only possible if $y \equiv 3 \pmod 6$. However, in this case $5^y + 1 \equiv 5^3 + 1 \equiv 0 \pmod 7$, so $5^y + 1$ is also divisible by 7, which is impossible.
Therefore we must have $t \leq 1$, which yields a (unique) solution $(x, y, z) = (2, 1, 2)$.

2. Since the quadrilaterals $BMPQ$ and $CNPQ$ are cyclic, we have $\angle BQN = \angle BQP + \angle PQN = \angle AMC + \angle MCA = 180° - \angle CAB$, so $ABQN$ is cyclic as well. Hence $\frac{\sin\angle BAQ}{\sin\angle NAQ} = \frac{BQ}{NQ}$. Moreover, triangles MBQ and CNQ are similar, so



$$\frac{\sin\angle BAQ}{\sin\angle CAQ} = \frac{BQ}{NQ} = \frac{BM}{CN} = \frac{AB}{AC}.$$

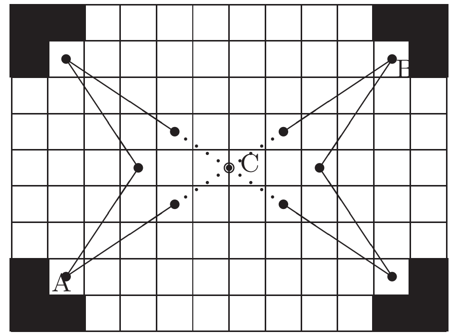
On the other hand, if AP meets BC at A_1, then by the Cheva theorem $\frac{BA_1}{A_1C} = \frac{BM}{MA} \cdot \frac{AN}{NC} = 1$, so A_1 is the midpoint of BC and

$$\frac{\sin\angle CAP}{\sin\angle BAP} = \frac{AB}{AC} \cdot \frac{AC \cdot AA_1 \sin\angle CAP}{AB \cdot AA_1 \sin\angle BAP} = \frac{AB}{AC} \cdot \frac{S_{\triangle CAA_1}}{S_{\triangle BAA_1}} = \frac{AB}{AC}.$$

Therefore, if we denote $\angle CAP = \varphi$, $\angle BAQ = \psi$ and $\angle BAC = \alpha$, we have $\frac{\sin\psi}{\sin(\alpha-\psi)} = \frac{\sin\varphi}{\sin(\alpha-\varphi)}$, which is equivalent to $\sin\psi\sin(\alpha-\varphi) = \sin\varphi\sin(\alpha-\psi)$. The addition formulas reduce the last equality to $0 = \sin\alpha(\sin\varphi\cos\psi - \sin\psi\cos\varphi) = \sin\alpha\sin(\varphi-\psi)$, from which we conclude that $\psi = \varphi$, as desired.

3. Place the given rectangle into the coordinate plane so that the center of the square at the intersection of i-th column and j-th row has the coordinates (i, j). Suppose that a desired numeration of the red points exists; it corresponds to a path, i.e. a closed poligonal line consisting of 96 segments of length $\sqrt{13}$, passing through each red point exactly once. Note that points (i, j) and (k, l) are adjacent in the path if and only if $\{|i-k|, |j-l|\} = \{2, 3\}$.

The center of symmetry must be at point $C(5\frac{1}{2},5)$. Consider the points $A(2,2)$, $B(11,8)$. These two points are symmetric with respect to C and divide the path into two parts γ_1 and γ_2. Note that, if the rectangular board is colored alternately white and black (like a chessboard), A and B are of different colors, and each segment connects two squares of different colors. It follows that each of γ_1, γ_2 consists of an odd number of segments. Thus these two parts are of different lengths and cannot be symmetric to each other. Therefore each of γ_1, γ_2 is centrally symmetric itself.



Being of an odd length, each of the parts γ_1, γ_2 must contain a segment which is centrally symmetric with respect to C. There are only two such segments - one connecting $(5,4)$ and $(8,6)$, and one connecting $(5,6)$ and $(8,4)$, so these two segments must be parts of our path. Moreover, point $(2,2)$ is connected with only two points, namely $(4,5)$ and $(5,4)$, so these three points are directly connected. Analogous conclusions can be made about points $(2,8)$, $(11,2)$ and $(11,8)$, so the closed path $(5,4)-(2,2)-(4,5)-(2,8)-(5,6)-(8,4)-(11,2)-(9,5)-(11,8)-(8,6)-(5,4)$ is entirely contained in our path, which is clearly a contradiction.

4. We start by observing that f is injective. From the known identity

$$(a^2+2b^2)(c^2+2d^2)=(ac\pm 2bd)^2+2(ad\mp bc)^2$$

we obtain $f(ac+2bd)^2+2f(ad-bc)^2=f(ac-2bd)^2+2f(ad+bc)^2$, assuming that the arguments are positive integers. Specially, for $b=c=d=1$ and $a\geq 3$ we have $f(a+2)^2+2f(a-1)^2=f(a-2)^2+2f(a+1)^2$. Denoting $g(n)=f(n)^2$ we get a recurrent relation $g(a+2)-2g(a+1)+2g(a-1)-g(a-2)=0$ whose characteristic polynomial is $(x+1)(x-1)^3$, which leads to

$$g(n)=A(-1)^n+B+Cn+Dn^2. \qquad (\dagger)$$

Substituting $m=n$ in the original equation yields $g(3g(n))=9n^4$, which together with $(\dagger)$ gives us

$$\begin{aligned} L=9n^4= &\quad A(-1)^{3(A(-1)^n+B+Cn+Dn^2)}+B+\underbrace{3C[A(-1)^n+B+Cn+Dn^2]} \\ &+\quad 9D[A(-1)^n+B+Cn+Dn^2]^2=R. \end{aligned}$$

Since $0=\lim_{n\to\infty}\frac{R-L}{n^4}=9D^3-9$, we have $D^3=1$ (so $D\neq 0$); similarly, $0=\lim_{n\to\infty}\frac{R-L}{n^3}=18D^2C$, so $C=0$. Moreover, for $n=2k$ and $n=2k+1$

respectively we obtain $0 = \lim_{k\to\infty} \frac{R-L}{(2k)^2} = 18D^2(A+B)$ and $0 = \lim_{k\to\infty} \frac{R-L}{(2k+1)^2} = 18D^2(-A+B)$, implying $A+B = -A+B = 0$; hence $A = B = 0$.

Finally, $g(n) = Dn^2$, $D^3 = 1$ and $g : \mathbb{N} \to \mathbb{N}$ gives us $D = 1$, i.e. $f(n) = n$. It is directly verified that this function satisfies the conditions of the problem.

Remark. Using limits can be avoided. Since the rigth-hand side in (†) takes only integer values, it follows that A, B, C, D are rational, so taking suitable multiples of integers for n eliminates the powers of -1 and leaves a polynomial equality.

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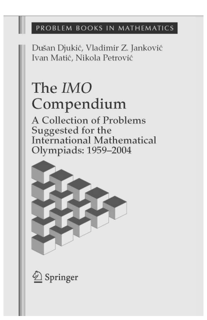

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