The 2nd Romanian Master of Mathematics Competition – Solutions
Bucharest, Saturday, February 28, 2009

Problem 1. For any positive integers $a_1, \dots, a_k$, let $n = \sum_{i=1}^{k} a_i$, and let $\binom{n}{a_1, \dots, a_k}$ be the multinomial coefficient $\frac{n!}{\prod_{i=1}^{k}(a_i!)}$. Let $d = \gcd(a_1, \dots, a_k)$ denote the greatest common divisor of $a_1, \dots, a_k$.

Prove that $\frac{d}{n}\binom{n}{a_1, \dots, a_k}$ is an integer.

Romania, Dan Schwarz[1]

Solution. The key idea is the fact that the greatest common divisor is a linear combination with integer coefficients of the numbers involved[2], i.e. there exist $u_i \in \mathbb{Z}$ such that $d = \sum_{i=1}^{k} u_i a_i$. But

$$\binom{n}{a_1, \dots, a_k} = \frac{n}{a_i}\binom{n-1}{a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_k},$$

so

$$\frac{d}{n}\binom{n}{a_1, \dots, a_k} = \sum_{i=1}^{k} u_i \binom{n-1}{a_1, \dots, a_{i-1}, a_i - 1, a_{i+1}, \dots, a_k},$$

which clearly is an integer, since multinomial coefficients are known (and easy to prove) to be integer. ■

Problem 2. A set S of points in space satisfies the property that all pairwise distances between points in S are distinct. Given that all points in S have integer coordinates (x, y, z), where $1 \le x, y, z \le n$, show that the number of points in S is less than $\min\left((n+2)\sqrt{n/3},\, n\sqrt{6}\right)$.

Romania, Dan Schwarz[3]

Solution. The critical idea is to estimate the total number possible T of distinct distances realized by pairs of points (x, y, z), of integer coordinates $1 \le x, y, z \le n$. However, any such distance is also realized by a pair anchored at $(1,1,1)$, from symmetry considerations.

But the number of distinct distances to points with no coordinates x, y, z equal is at most $\binom{n}{3} = \frac{1}{6}n(n-1)(n-2)$; the number of distinct distances to points with two of the three coordinates x, y, z equal is at most $2\binom{n}{2} = n(n-1)$; while the number of distinct distances to points with all three coordinates x, y, z equal is $n-1$, hence

$$T \le \frac{1}{6}n(n-1)(n-2) + n(n-1) + (n-1) < \frac{1}{6}(n^3 + 3n^2 + 2n).$$

On the other hand, the total number of distinct distances between the N points in S needs be $\binom{N}{2} = \frac{1}{2}N(N-1) \le T$, yielding

$$(2N-1)^2 < \frac{1}{3}(4n^3 + 12n^2 + 8n) + 1 \le \frac{1}{3}(2n\sqrt{n} + 3\sqrt{n})^2,$$

hence $N < \frac{1}{2}\left((2n+3)\sqrt{n/3} + 1\right) \le (n+2)\sqrt{n/3}$ for $n \ge 3$. One can easily check that the inequality is true for $n = 2$ also, since then[4] $T = 3$.

On the other hand, since the squares of the distances can only take the integer values between 1 and the trivial upper bound $3(n-1)^2$ (for the diagonal of the cube), it follows that $T \le 3(n-1)^2$, yielding $N < n\sqrt{6}$. ■

Problem 3. Given four points A_1, A_2, A_3, A_4 in the plane, no three collinear, such that

$$A_1A_2 \cdot A_3A_4 = A_1A_3 \cdot A_2A_4 = A_1A_4 \cdot A_2A_3,$$

let us denote by O_i the circumcenter of $\Delta A_jA_kA_\ell$, with $\{i, j, k, \ell\} = \{1,2,3,4\}$.

Assuming $A_i \ne O_i$ for all indices i, prove that the four lines A_iO_i are concurrent or parallel.

Bulgaria, Nikolai Ivanov Beluhov

Solution. (D. Schwarz) The given triple equality being invariated by any permutation in $\mathscr{S}_4$, it is enough to prove that the lines A_iO_i for $2 \le i \le 4$ are concurrent or parallel. The relations can then be written

$$\frac{A_1A_2}{A_1A_3} = \frac{A_4A_2}{A_4A_3}, \quad \frac{A_1A_3}{A_1A_4} = \frac{A_2A_3}{A_2A_4}, \quad \frac{A_1A_4}{A_1A_2} = \frac{A_3A_4}{A_3A_2}.$$

Consider the Apollonius circles Γ_k of centers $\omega_k \in A_iA_j$, for $\{i, j, k\} = \{2,3,4\}$, determined by the point A_1, which therefore lies on all three, while the points A_k lie on Γ_k. Moreover, the points ω_k are collinear, since the point A'_k which is the other meeting point (than A_1, if any) of Γ_i and Γ_j fulfills

$$\frac{A'_kA_j}{A'_kA_k} = \frac{A_iA_j}{A_iA_k} \text{ and } \frac{A'_kA_i}{A'_kA_k} = \frac{A_jA_i}{A_jA_k}, \text{ thus } \frac{A'_kA_i}{A'_kA_j} = \frac{A_kA_i}{A_kA_j},$$

therefore A'_k also lies on Γ_k, hence all three circles Γ_k share the same meeting point(s), thus their centers are collinear.

Now, the circumcenters O_i and O_j, as well as the point ω_k, lie on the perpendicular bisector of the segment A_1A_k, for $\{i, j, k\} = \{2,3,4\}$. It follows that the pairs of lines A_iA_j, O_iO_j meet at the collinear points ω_k. Desargues' theorem for the perspective triangles $\Delta A_iA_jA_k$ and $\Delta O_iO_jO_k$ yields the claim. ■

Alternate Solution. The author's original solution makes use of inversions of poles A_i to reach the same conclusion via Desargues, in a dual-by-inversion to the solution above manner, with a lot more details than concepts. We feel that making use of the well-known properties of the Apollonius circles renders the idea in a more striking way. ■

Remark. There exists a particular (degenerate) case, when the points are the vertices of a kite of $\frac{\pi}{6}$ equal angles, hence one of the associated ratios is 1, so a corresponding Apollonius circle degenerates to the perpendicular bisector. This (together with the use of Desargues) shows the deep projective nature of the problem, better handled through projective methods.

Also, there is no converse implication, since the case of concyclic points trivially warrants the conclusion, without fulfilling the stated condition (as in conflict with Ptolemy's relation).

Problem 4. For a finite set X of positive integers, let

$$\Sigma(X) = \sum_{x\in X} \arctan \frac{1}{x}.$$

Given a finite set S of positive integers for which $\Sigma(S) < \frac{\pi}{2}$, show that there exists at least one finite set T of positive integers for which $S \subset T$ and $\Sigma(T) = \frac{\pi}{2}$.

United Kingdom, Kevin Buzzard

Solution. (D. Schwarz) We will step-by-step augment the set S with positive integers t_n, by taking each time t_n as the least positive integer larger than $\max(S)$, and not already used, such that $\Sigma(S \cup \{t_1, t_2, \dots, t_n\})$ remains at most $\frac{\pi}{2}$ (this is possible since $\arctan\frac{1}{t} \to 0$ when $t \to \infty$). If at some point we get exactly $\frac{\pi}{2}$ we are through, since we have augmented S to a set T as required, so assume the process continues indefinitely. Clearly the sequence $(t_n)_{n\ge1}$ is built (strictly) increasing, so for all $n \ge 1$ we have $t_{n+1} > t_n > \max(S)$.

We will make some useful notations. Take $S_0 = S$, $S_{n+1} = S_n \cup \{t_{n+1}\}$, for $n \in \mathbb{N}$. Also take $x_n = \tan\left(\frac{\pi}{2} - \Sigma(S_n)\right)$. Using the well-known formula $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ one can easily prove by simple induction that a lesser than $\frac{\pi}{2}$ sum of arcs of rational tangents is as well an arc of rational tangent, therefore $x_n = \frac{p_n}{q_n}$, with $p_n, q_n \in \mathbb{N}^*$, $(p_n, q_n) = 1$. Since arctan is increasing, we need take $t_{n+1} \ge \left\lceil \frac{1}{x_n} \right\rceil$ in order that we may augment S_n with t_{n+1} to obtain S_{n+1}.

Assume that for all $n \ge 1$ we have $\frac{1}{x_n} \le t_n$. Since we need both $t_{n+1} \ge \left\lceil \frac{1}{x_n} \right\rceil$ and $t_{n+1} > t_n \ge \frac{1}{x_n}$, it follows that $t_{n+1} = t_n + 1$ (the least available value), so $t_{k+1} = t_1 + k$ for all $k \ge 0$. But then $\dfrac{\pi}{2} > \Sigma(\{t_1, t_2, \dots, t_n\}) = \sum_{k=0}^{n-1} \arctan \dfrac{1}{t_1 + k} > \dfrac{1}{2}\sum_{k=0}^{n-1} \dfrac{1}{t_1+k} \to \infty$ when $n \to \infty$, absurd (see *Lemma*).

Therefore there exists some $N \ge 1$ for which $\frac{1}{x_N} > t_N$, so $\left\lceil \frac{1}{x_N} \right\rceil$ is available for t_{N+1}. Moreover, for any $n \ge N$ with $t_{n+1} = \left\lceil \frac{1}{x_n} \right\rceil$, we have $x_{n+1} = \dfrac{x_n - \frac{1}{t_{n+1}}}{1 + x_n \frac{1}{t_{n+1}}} = \dfrac{x_n t_{n+1} - 1}{t_{n+1} + x_n} < \dfrac{x_n}{t_{n+1} + x_n} < \dfrac{1}{t_{n+1}}$, since $t_{n+1} = \left\lceil \frac{1}{x_n} \right\rceil$ implies $x_n t_{n+1} - 1 < x_n$; and so we can take $t_{n+1} = \left\lceil \frac{1}{x_n} \right\rceil$ indefinitely for $n \ge N$. Now we use the fact that $x_n = \frac{p_n}{q_n}$.

Then $\dfrac{p_{n+1}}{q_{n+1}} = \dfrac{\frac{p_n}{q_n} - \frac{1}{t_{n+1}}}{1 + \frac{p_n}{q_n}\frac{1}{t_{n+1}}} = \dfrac{p_n t_{n+1} - q_n}{q_n t_{n+1} + p_n}$, hence $p_{n+1} \le p_n t_{n+1} - q_n < p_n$, since $t_{n+1} = \left\lceil \dfrac{q_n}{p_n} \right\rceil$, and so $t_{n+1} < \dfrac{q_n}{p_n} + 1$.

Therefore the sequence $(p_n)_{n\ge1}$ of the numerators of x_n eventually becomes (strictly) decreasing, absurd for any sequence of positive integers. ■

Lemma. For $x \in (0, \frac{\pi}{2})$ one has $\arctan x > \frac{x}{2}$.

Proof. We start by proving that under given condition one has $\sin x > \tan\frac{x}{2}$, in turn equivalent to $2\sin\dfrac{x}{2}\cos\dfrac{x}{2} > \dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}}$, $2\cos^2\frac{x}{2} - 1 > 0$, and finally $\cos x > 0$, patently true.

Now, arctan is increasing, hence applied to the above, together with the well-known inequality $x > \sin x$, true for all $x > 0$, yields $\arctan x > \arctan\sin x > \arctan\tan\frac{x}{2} = \frac{x}{2}$. □

As a corollary, $\arctan\frac{1}{n} > \frac{1}{2n}$, for all positive integers n, inequality used to yield the divergence of the series $\sum_{n\ge1} \arctan \dfrac{1}{n}$ in the above solution.

Remark. The above solution shows that it is irrelevant that we start with the arc $\dfrac{\pi}{2} - \sum_{s\in S} \arctan\dfrac{1}{s}$; in fact we may state the problem like this

> *Prove that for any arc $\alpha \in (0, \frac{\pi}{2})$ of some rational tangent $\tau = \tan\alpha$, and any finite set S of distinct positive integers, there exists some finite set T of distinct positive integers such that $T \cap S = \emptyset$ and*
>
> $$\sum_{t\in T} \arctan\frac{1}{t} = \alpha.$$

The problem is strongly reminiscent of a strengthened form of the famous *Egyptian fraction*[5] theorem

> *Prove that for any rational number $r \in (0,1)$, and any finite set S of distinct positive integers, there exists a finite set T of distinct positive integers such that $T \cap S = \emptyset$ and*
>
> $$\sum_{t\in T} \frac{1}{t} = r.$$

All the ingredients are there: the greedy algorithm, going beyond the largest element of S, using the divergence of the series $\sum_{n\ge1} \dfrac{1}{n}$, and the (Fermat) infinite descent method of a (strictly) decreasing sequence of positive integers.

In fact, it is enough to consider a (strictly) increasing function $f : \mathbb{Q}_+ \to \mathbb{R}_+$ with the properties that there exists a function $\varphi : \mathbb{Q}_+ \times \mathbb{Q}_+ \to \mathbb{Q}_+$ such that $f(r) - f(s) = f(\varphi(r,s))$ for any $0 \le s < r$ in $\mathbb{Q}$, $\lim_{x\to0} f(x) = 0$, and $\lim_{n\to\infty} \sum_{k=1}^{n} f\left(\dfrac{1}{k}\right) = \infty$.

Moreover, we need that $\varphi(r,s)$ has not larger numerator than $r - s$, and not lesser denominator. Then the Egyptian fraction method extends perfectly. Or $f(x) = \arctan x$ and $\varphi(x,y) = \frac{x-y}{1+xy}$ conform to this model. **END**

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[1] Based on a property of quasi-Catalan numbers of J. Conway, see [GUY, R.K., *Unsolved Problems in Number Theory*].

[2] Easily proven by induction from the classical Bézout's relation $\gcd(M,N) = uM + vN$ for some integers u, v.

[3] A 3-dimensional extrapolation of a plane lattice points case study of P. Erdös and R.K. Guy.

[4] An example of $N = 3$ points for $n = 2$ is $(1,1,1)$, $(2,2,1)$, $(2,2,2)$; and of $N = 4$ points for $n = 3$ is $(1,1,1)$, $(1,1,2)$, $(2,2,1)$, $(2,3,3)$.

[5] An Egyptian fraction is written as a finite sum of fractions with all unit numerators and all distinct denominators. Such fractions were used by ancient Egyptians, as apparent in the Rhind Papyrus, but their use is discontinued today.