26-th Balkan Mathematical Olympiad

Kragujevac, Serbia – April 30, 2009

1. Find all integer solutions of the equation

$$3^x - 5^y = z^2.$$

(Greece)

2. In a triangle ABC, points M and N on the sides AB and AC respectively are such that $MN \parallel BC$. Let BN and CM intersect at point P. The circumcircles of triangles BMP and CNP intersect at two distinct points P and Q. Prove that $\angle BAQ = \angle CAP$. *(Moldova)*

3. A 9×12 rectangle is divided into unit squares. The centers of all the unit squares, except the four corner squares and the eight squares adjacent (by side) to them, are colored red. Is it possible to numerate the red centers by $C_1, C_2, \ldots, C_{96}$ so that the following two conditions are fulfilled:

1° All segments $C_1C_2, C_2C_3, \ldots C_{95}C_{96}, C_{96}C_1$ have the length $\sqrt{13}$;

2° The polygonal line $C_1C_2 \ldots C_{96}C_1$ is centrally symmetric? *(Bulgaria)*

4. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$f\left(f(m)^2 + 2f(n)^2\right) = m^2 + 2n^2 \qquad \text{for all } m, n \in \mathbb{N}.$$

(Bulgaria)

Each problem is worth 10 points.
Time allowed: $4\frac{1}{2}$ hours.