Serbian Mathematical Olympiad 2008

for high school students

Belgrade, April 12–13, 2008

Problems and Solutions

SHORT HISTORY AND SYSTEM

Mathematical competitions in Serbia have been held since 1958. In the first few years only republic competitions within the former Yugoslavia, which Serbia was a part of, were held. The first Federal Mathematical Competition in Yugoslavia was held in Belgrade in 1960, and since then it has been held regularly every year, skipping only 1999 for a non-mathematical reason we prefer not to talk about here. The system has suffered relatively few changes. The earliest Federal Competitions were organized for 3rd and 4th grades of high school only; 2nd grade was added in 1970, and 1st grade in 1974. Since 1982, 3rd and 4th grades compose a single category. After the breakdown of the old Yugoslavia in 1991, the entire system was continued in the newly formed FR Yugoslavia, later renamed Serbia and Montenegro. The separation of Montenegro finally made the federal competition senseless as such. Thus, starting with 2007, the federal competition and team selection exam are replaced by a two-day Serbian Mathematical Olympiad.

Today a mathematical competition season in Serbia consists of four rounds of increasing difficulty:

- Municipal round, held in early February. The contest consists of 5 problems, each 20 points worth, to be solved in 3 hours. Students who perform well qualify for the next round (50-60 points are usually enough).
- Regional round, held in late February in the same format as the municipal round. Each student's score is added to that from the municipal round. Although the number of students qualified for the state round is bounded by regional quotas, a total score of 110-120 should suffice.
- State (republic) round, held in late March in a selected town in the country. There are roughly 200 to 300 participants. The contest consists of 5 problems in 4 hours.
- Serbian Mathematical Olympiad (SMO), held in middle April in a selected town in the country. The participants are selected through the state round: 26 from A category (distribution among grades: 3+5+8+10), 3 from B category (0+0+1+2), plus those members of the last year's olympic team who did not manage to qualify otherwise. Six most successful contestants are invited to the olympic team.

Since 1998, the problems for each grade on the three preliminary rounds are divided into A category (specialized schools and classes) and B category (others). A student from B category is normally allowed to work the problems for A category. On the SMO, all participants do the same problems.

The Serbian Mathematical Olympiad 2008 for high school students was held in Belgrade on April 12–13. There were 27 students from Serbia and 4 guest students from the Serb entity of Bosnia and Herzegovina taking part on the competition. The contest turned out to be quite difficult and as many as three of the chosen problems (namely, problems 2, 3 and 6) were only partially solved by the students. Based on the results of the competition the team of Serbia for the 25-th Balkan Mathematical Olympiad and the 49-th International Mathematical Olympiad was selected:

Dušan Milijančević	Math High School, Belgrade	25 points
Luka Milićević	Math High School, Belgrade	18 points
Aleksandar Vasiljković	Math High School, Belgrade	16 points
Teodor von Burg	Math High School, Belgrade	13 points
Vladimir Nikolić	Math High School, Belgrade	13 points
Marija Jelić	Math High School, Belgrade	12 points

In this booklet we present the problems and full solutions of the Serbian Mathematical Olympiad.

SERBIAN MATHEMATICAL OLYMPIAD

for high school students

Belgrade , 12.04.2008.

First Day

1. Solve in integers the equation

$$12^x + y^4 = 2008^z.$$

(Miloš Milosavljević)

2. Given a triangle ABC, let D and E be the points on line AB such that $D-A-B-E$, $AD = AC$ and $BE = BC$. The bisectors of the angles at A and B meet the opposite sides of the triangle at P and Q respectively, and meet the circumcircle at M and N, respectively. The line joining A with the circumcenter of triangle BME and the line joining B with the circumcenter of triangle AND intersect at point X, $X \neq C$. Prove that $CX \perp PQ$. *(Dušan Djukić)*

3. If a, b and c are arbitrary positive numbers with $a + b + c = 1$, prove the inequality

$$\frac{1}{bc + a + \frac{1}{a}} + \frac{1}{ca + b + \frac{1}{b}} + \frac{1}{ab + c + \frac{1}{c}} \leqslant \frac{27}{31}.$$

(Marko Radovanović with collaborators)

Time allowed: 270 minutes.
Each problem is worth 7 points.

SERBIAN MATHEMATICAL OLYMPIAD

for high school students

Belgrade, 13.04.2008.

Second Day

Each point of a plane is painted in one of three colors. Show that there exists a triangle such that:

1° all three vertices of the triangle are of the same color;

2° the radius of the circumcircle of the triangle is 2008;

3° one angle of the triangle is either two or three times greater than one of the other two angles. *(Vladimir Baltić)*

The sequence $(a_n)_{n\geq 1}$ is defined by $a_1 = 3$, $a_2 = 11$, and $a_n = 4a_{n-1} - a_{n-2}$ for $n \geq 3$. Prove that each term of this sequence is of the form $a^2 + 2b^2$ for some natural numbers a and b. *(Djordje Baralić)*

Let $ABCDE$ be a convex pentagon in which $AB = 1$, $\angle BAE = \angle ABC = 120°$, $\angle CDE = 60°$ and $\angle ADB = 30°$. Prove that the area of pentagon $ABCDE$ is less than $\sqrt{3}$. *(Miloš Milosavljević)*

Time allowed: 270 minutes.
Each problem is worth 7 points.

SOLUTIONS

1. For $x<0$ or $z\le 0$ the only solution is the trivial $(0,0,0)$. From now on we assume $z>0$. Since $2008=2^3\cdot 251$, both sides of the equation are divisible by 251. Suppose that $x=2x_1$ is even. Then $(2^{x_1})^2\equiv -(y^2)^2 \pmod{251}$, which upon raising to the 125-th power yields $1\equiv (2^{x_1})^{250}\equiv -(y^2)^{250}\equiv -1$ by the Fermat theorem, which is impossible. Hence x must be odd.

Obviously, y is even. Write $y=2^u y_1$ for some odd y_1. Then we have

$$2^{2x}3^x+2^{4u}y_1^4=2^{3z}251^z.$$

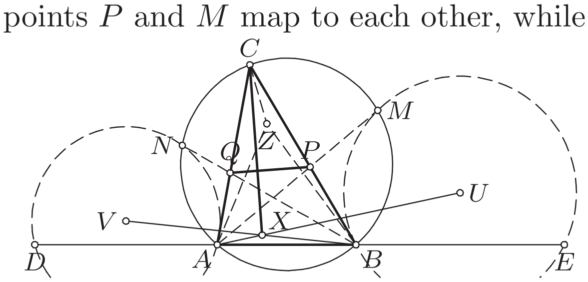
Since $2x\ne 4u$ for x odd, the largest power of two in the left hand side is either 2^{2x} or 2^{4u}, whereas the largest power of two in the right hand side is 2^{3z}, implying that either $3z=2x$ or $3z=4u$. We shall show that none of these two cases admits a solution.

(i) $3z=2x<4u$; hence $2\mid z$. Canceling 2^{2x} yields $3^x+2^{4u-2x}y_1^4=251^z$ which is impossible because the left hand side is of the form $4k+3$ (because $2\nmid x$), while the other side is of the form $4k+1$ ($k\in\mathbb{N}$).

(ii) $3z=4u<2x$; again, $2\mid z$. Canceling 2^{4u} yields $2^{2x-4u}3^x+y_1^4=251^z$. The right hand side is of the form $5k+1$, so for $5\nmid y_1$ we have $y_1^4\equiv 1$ and $2^{2x-4u}3^x\equiv 0 \pmod 5$ which is impossible, while for $5\mid y_1$ we get $1\equiv 2^{2x-4u}3^x\equiv \pm 3^x\equiv \pm 3 \pmod 5$ jer $2\nmid x$, impossible again.

Second solution. The left hand side is of the form a^2+b^2 for even x, and of the form a^2+3b^2 for odd x. However, since both -1 and -3 are quadratic non-residues modulo 251, neither a^2+b^2 nor a^2+3b^2 can be divisible by 251 unless $251\mid a$. Therefore, the equation has no integral solutions for $z>0$.

2. Denote by U the circumcenter of $\triangle BME$. Apply the inversion with center A and power $AB\cdot AC$. Points B and C map to points B' and C' symmetric to points C and B respectively with respect to AP, points P and M map to each other, while E maps to the reflection E' of point Q in AP. Therefore, line AU coincides with the line joining A with the circumcenter of $\triangle B'PE'$ (note that the centers do not map to each other!). We notice that this line is the reflection of the line AZ in the angle bisector at A, where Z is



the circumcenter of $\triangle CPQ$.

Analogously, the line joining B with the center V of circle AND is the reflection of BZ in the angle bisector at B. By the trigonometric Ceva theorem, the reflections of the lines AU, BV, CX in the angle bisectors at A, B, C respectively are concurrent (at the isogonal conjugate of X with respect to $\triangle ABC$), which implies that line CZ is symmetric to CX with respect to the angle bisector at C. But Z is the circumcenter of triangle CPQ, hence the line CX contains the altitude of triangle CPQ, as we wanted to show.

The given inequality is equivalent to the inequality

$$\frac{a}{p+a^2}+\frac{b}{p+b^2}+\frac{c}{p+c^2}\leq\frac{27}{31},$$

where $a+b+c=1$ and $p=abc+1$. Consider the function

$$f(x)=\frac{3(a+b+c)}{3x+a^2+b^2+c^2}-\frac{a}{x+a^2}-\frac{b}{x+b^2}-\frac{c}{x+c^2}.$$

We shall prove that $f(x)\geq 0$ for all $x\geq ab+bc+ca$. Reducing the expression for $f(x)$ to the common denominator gives us

$$f(x)=\frac{Ax^2+Bx+C}{(x+a^2)(x+b^2)(x+c^2)(3x+a^2+b^2+c^2)},$$

where $A\geq 0\geq C$. In fact, one easily finds

$$A=2a^3+2b^3+2c^3-ab(a+b)-ac(a+c)-bc(b+c)\geq 0,$$
$$C=-abc[a(b^3+c^3)+b(c^3+a^3)+c(a^3+b^3)-2abc(a+b+c)]\leq 0.$$

Note that we do not need the exact expression for B. Thus, the polynomial $P(x)=Ax^2+Bx+C$ (unless it is identically 0) has two real zeros, one being positive (say, $x=x_0$) and the other negative, and satisfies $P(x)\leq 0$ for $0\leq x\leq x_0$ and $P(x)\geq 0$ for $x\geq x_0$. We claim that $f(ab+bc+ca)\geq 0$. Indeed,

$$\begin{aligned}&f(ab+bc+ca)\\&=\frac{3(a+b+c)}{a^2+b^2+c^2+3(ab+bc+ca)}-\frac{a}{(a+b)(a+c)}-\frac{b}{(b+c)(b+a)}-\frac{c}{(c+a)(c+b)}\\&=\frac{3(a+b+c)}{a^2+b^2+c^2+3(ab+bc+ca)}-\frac{2(ab+bc+ca)}{(a+b)(b+c)(c+a)}\geq 0\qquad\text{jer je}\end{aligned}$$

$$\frac{3(a+b+c)}{a^2+b^2+c^2+3(ab+bc+ca)}\geq\frac{9}{4(a+b+c)}\geq\frac{2(ab+bc+ca)}{(a+b)(b+c)(c+a)},$$

as we claimed. Therefore $P(ab+bc+ca) \geq 0$, i.e. $x_0 \leq ab+bc+ca$, which implies that $P(x) \geq 0$ and $f(x) \geq 0$ for all $x \geq ab+bc+ca$. In particular, $f(1+abc) \geq 0$ holds because $1+abc > 1 > ab+bc+ca$. Thus we have proved that

$$\frac{a}{1+abc+a^2} + \frac{b}{1+abc+b^2} + \frac{c}{1+abc+c^2} \leq \frac{3}{3+a^2+b^2+c^2+3abc}. \qquad (1)$$

It remains to show that $a^2+b^2+c^2+3abc \geq \frac{4}{9}$, which will, combined with (1), give the desired inequality. Homogenization yields $9(a+b+c)(a^2+b^2+c^2)+27abc \geq 4(a+b+c)^3$, which is equivalent to

$$5(a^3+b^3+c^3)+3abc \geq 3(ab(a+b)+ac(a+c)+bc(b+c)).$$

The last inequality is an immediate consequence of the Schur and Muirhead inequalities. This completes our proof.

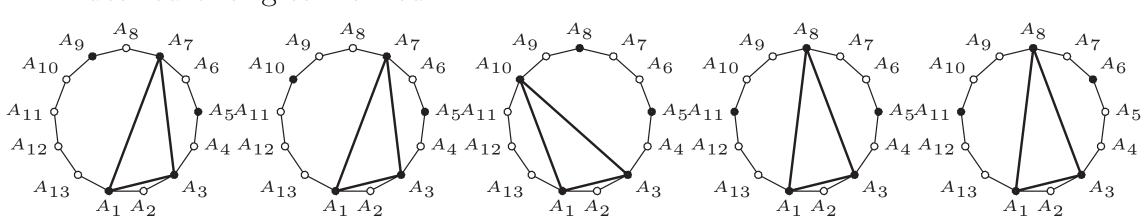
Second solution. The given inequality is equivalent to the following symmetric inequality which is a direct consequence of the Muirhead inequality:

$$\frac{23}{2}T_{900} + 122T_{810} + 260T_{720} + 282T_{630} + 193T_{540} + \frac{547}{2}T_{711} + 807T_{620} + 284T_{531}$$
$$+91T_{522} - 98T_{441} - 1669T_{432} - 557T_{333} \geq 0,$$

where T_{ijk} denotes the symmetric sum $x^i y^j z^k + \cdots$.

4. Consider a regular 13-gon $A_1A_2\ldots A_{13}$ inscribed in a circle of radius 2008. By the pigeon-hole principle, five of its vertices are of the same color (e.g. red). We distinguish two cases.

(i) No two of the red vertices are adjacent. Each distribution of the red points (up to a rotation) corresponds to a decomposition of number 13 into 5 summands greater than 1. There are five such distributions, as shown below with the desired triangles marked.



2 + 2 + 2 + 2 + 5   2 + 2 + 2 + 3 + 4   2 + 2 + 3 + 2 + 4   2 + 2 + 3 + 3 + 3   2 + 3 + 2 + 3 + 3

(ii) Some two red vertices are adjacent, say A_1 and A_2. If any of the points $A_4, A_5, A_6, A_{10}, A_{11}, A_{12}$ is red, it will determine a desired triangle with the vertices A_1, A_2. From now on we assume that none of these six points is red.

Then at least three of the points $A_3, A_7, A_8, A_9, A_{13}$ are red. If A_3 is among them (analogously for A_{13}), then at least one of A_7, A_9, A_{13} is red, and this point with the points A_1 and A_3 determines a desired triangle. The only remaining case is when A_7, A_8, A_9 are red, but then $\triangle A_1A_7A_9$ satisfies the conditions.

We have $a_1 = 1 + 2\cdot 1^2$, $a_2 = 3^2 + 2\cdot 1^2$, $a_3 = 3^2 + 2\cdot 4^2$, $a_4 = 11^2 + 2\cdot 4^2$, etc. We prove by induction on n that

$$a_{2n-1} = a_{n-1}^2 + 2\left(\frac{a_n - a_{n-1}}{2}\right)^2 \quad \text{and} \quad a_{2n} = a_n^2 + 2\left(\frac{a_n - a_{n-1}}{2}\right)^2,$$

where $a_0 = 1$. Assuming this statement for n, we have

$$\begin{aligned}
a_{2n+1} &= 4a_{2n} - a_{2n-1} = 4a_n^2 + 8\left(\tfrac{a_n - a_{n-1}}{2}\right)^2 - a_{n-1}^2 - 2\left(\tfrac{a_n - a_{n-1}}{2}\right)^2 \\
&= \tfrac{11}{2}a_n^2 - 3a_na_{n-1} + \tfrac{1}{2}a_{n-1}^2 = \tfrac{11}{2}a_n^2 - 3a_n(4a_n - a_{n+1}) + \tfrac{1}{2}(4a_n - a_{n+1})^2 \\
&= \tfrac{3}{2}a_n^2 - a_na_{n+1} + \tfrac{1}{2}a_{n+1}^2 = a_n^2 + 2\left(\tfrac{a_{n+1} - a_n}{2}\right)^2; \\
a_{2n+2} &= 4a_{2n+1} - a_{2n} = 4a_n^2 + 8\left(\tfrac{a_{n+1} - a_n}{2}\right)^2 - a_n^2 - 2\left(\tfrac{a_n - a_{n-1}}{2}\right)^2 \\
&= 3a_n^2 + 8\left(\tfrac{a_{n+1} - a_n}{2}\right)^2 - 2\left(\tfrac{a_{n+1} - 3a_n}{2}\right)^2 = \tfrac{3}{2}a_{n+1}^2 - a_na_{n+1} + \tfrac{1}{2}a_n^2 \\
&= a_{n+1}^2 + 2\left(\tfrac{a_{n+1} - a_n}{2}\right)^2,
\end{aligned}$$

which finishes the proof.

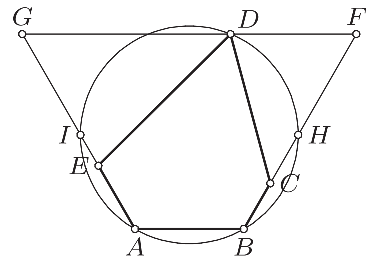
Second solution. It is known that an odd integer $m > 1$ can be represented as $a^2 + 2b^2$ for some co-prime $a, b \in \mathbb{N}$ if and only if all prime divisors of m are of the form $8k+1$ or $8k+3$ for $k \in \mathbb{N}_0$. It is easy to see that each a_n is odd; It remains to show that if a prime number p divides a_n, then $p = 8k+1$ or $8k+3$ for some $k \in \mathbb{N}_0$.

An easy induction on n shows that $a_na_{n+2} = a_{n+1}^2 + 2$. Indeed, this holds for $n \le 2$, while for $n > 2$, assuming the statement for $n-2$, we obtain

$$\begin{aligned}
\tfrac{a_{n+1}^2 + 2}{a_n} &= \tfrac{(4a_n - a_{n-1})^2 + 2}{a_n} = 16a_n - 8a_{n-1} + \tfrac{a_{n-1}^2 + 2}{a_n} \\
&= 16a_n - 8a_{n-1} + a_{n-2} = 4a_{n+1} - a_n = a_{n+2}.
\end{aligned}$$

It follows that -2 is a quadratic residue modulo each prime divisor p of a_n, so $p \equiv 1$ or $p \equiv 3 \pmod 8$.

Let c be the circumcircle of triangle ABD, and l be the line through D parallel to



AB. The radius of circle c is 1. The rays BC and AE meet c at points H and I, and meet the line l at F and G, respectively. Triangles FCD and GDE are similar because $\angle CFD = \angle DGE = 60^\circ$ and $\angle FCD = 120^\circ - \angle CDF = \angle GDE$. Denote by $k = \frac{FC}{GD} = \frac{FD}{GE}$ the similitude and by h the distance of D from line HI, and $x = FD$, $y = GD$. It is easy to obtain $x+y = 2 + \frac{2}{\sqrt{3}}h$ and $xy = \frac{4}{3}h^2 + \frac{2}{\sqrt{3}}h$ (the product xy is the power of point F with respect to c and equals $OF^2 - 1$, where O is the center of circle c). Thus we get

$$P_{ABFG} = \tfrac{1}{2}(1+x+y)\left(\tfrac{\sqrt{3}}{2}+h\right) = \tfrac{1}{\sqrt{3}}h^2 + 2h + \tfrac{3\sqrt{3}}{4}$$
$$P_{FCD} + P_{GDE} = \tfrac{1}{2}(x \cdot FC + y \cdot GE)\sin 60^\circ = \tfrac{\sqrt{3}}{4}xy\left(k+\tfrac{1}{k}\right) \geq \tfrac{\sqrt{3}}{2}xy = \tfrac{2}{\sqrt{3}}h^2 + h;$$

hence

$$P_{ABCDE} = P_{ABFG} - (P_{FCD} + P_{GDE}) \leq -\frac{1}{\sqrt{3}}h^2 + h + \frac{3\sqrt{3}}{4} = f(h).$$

The quadratic function $f(h)$ attains its maximum for $h = \frac{\sqrt{3}}{2}$, which proves that $P_{ABCDE} \leq \sqrt{3}$. The equality would be attained only if $h = \frac{\sqrt{3}}{2}$ and $k = 1$. Then (assuming without loss of generality that $DA \geq DB$) points D and B would be symmetric with respect to HI, so the triangle ADG would be equilateral and $FC = GD = GA = FB$ which is impossible because B and C would coincide. Thus the above inequality is strict.

Second solution. Suppose that triangle ABD is not obtuse-angled. Then the reflections C' and E' of the points C and E in the lines BD and AD respectively lie within the triangle ABD, on the same line through D, and

$$S_{ABCDE} = S_{ABD} + S_{ADE} + S_{BDC} = S_{ABD} + S_{ADE'} + S_{BDC'} \leq 2S_{ABD} - S_{ABF},$$

where F is the intersection of lines AE' and BC'. Equality holds if and only if $F \equiv C' \equiv E'$. Denote $\angle BAD = \alpha$. Then $\angle ABD = 150^\circ - \alpha$, $\angle BAF = 2\alpha - 120^\circ$, $\angle ABF = 180^\circ - 2\alpha$, so we have

$$S_{ABD} = \sin\alpha \sin(150^\circ - \alpha) = \tfrac{\sqrt{3}}{4} + \tfrac{1}{2}\cos(150^\circ - 2\alpha) = \tfrac{\sqrt{3}}{4} + \tfrac{1}{2}u,$$
$$S_{ABF} = \tfrac{1}{\sqrt{3}}\sin(2\alpha - 120^\circ)\sin 2\alpha = -\tfrac{\sqrt{3}}{12} + \tfrac{\sqrt{3}}{6}\cos(300^\circ - 4\alpha) = -\tfrac{\sqrt{3}}{4} + \tfrac{1}{\sqrt{3}}u^2,$$

where $\cos(150^\circ - 2\alpha) = u$ and therefore $\cos(300^\circ - 4\alpha) = 2u^2 - 1$. Now we have

$$S_{ABCDE} \leq 2S_{ABD} - S_{ABF} = \frac{3\sqrt{3}}{4} + u - \frac{u^2}{\sqrt{3}} \leq \sqrt{3},$$

with a possible equality case for $u = \frac{\sqrt{3}}{2}$, i.e. $\alpha \in \{60°, 90°\}$, and $F \equiv C' \equiv E'$, but equality is never attained because for these values of α point F lies at the right angle vertex, making the pentagon degenerate.

In the case of an obtuse-angled triangle ABD, with the same notation, point F lies outside triangle ABD, but the above formula for the area of $\triangle ABF$ takes a negative value, and we again obtain $S_{ABCDE} \leq \frac{3\sqrt{3}}{4} + u - \frac{u^2}{\sqrt{3}} < \sqrt{3}$.

Mathematical Competitions in Serbia
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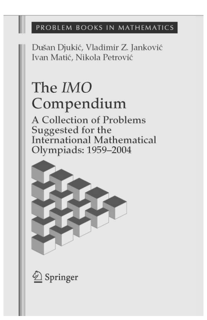

The IMO Compendium

This book attempts to gather all the problems appearing on the IMO, as well as the so-called *short-lists* from 35 years, a total of 864 problems, all of which are solved, often in more than one way. The book also contains 1036 problems from various *long-lists* over the years, for a grand total of 1900 problems.
In short, The IMO Compendium is an invaluable resource, not only for high-school students preparing for mathematical competitions, but for anyone who loves and appreciates math.

Publisher: Springer (2006); Hardcover, 746 pages; Language: English; ISBN: 0387242996

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