Romanian Master in Mathematics

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Problem 1. Let ABC be an equilateral triangle. P is a variable point internal to the triangle and its perpendicular distances to the sides are denoted by a^2, b^2 and c^2 for positive real numbers a, b and c. Find the locus of points P so that a, b and c can be the sides of a non-degenerate triangle.

Problem 2. Prove that any bijective function $f : \mathbb{Z} \to \mathbb{Z}$ can be written as $f = u + v$, where $u, v : \mathbb{Z} \to \mathbb{Z}$ are bijective functions.

Problem 3. Given a positive integer $a > 1$, prove that any positive integer N has a multiple in the sequence

$$(a_n)_{n\geqslant 1}, \quad a_n = \left\lfloor \frac{a^n}{n} \right\rfloor .$$

Problem 4. Consider a square of side length a positive integer n. Suppose that there are $(n+1)^2$ points in the interior of the square. Show that three of these points define a (possibly degenerate) triangle of area at most $\frac{1}{2}$.

Every problem is worth 7 points.
Time allowed is 5 hours.