SERBIAN MATHEMATICAL OLYMPIAD

for high school students

First Day

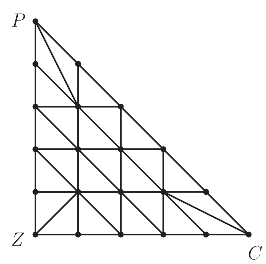
Belgrade , April 2, 2007

1. Let D be the point on side AC of a triangle ABC with $AB < BC$ such that $AB = BD$. The incircle of $\triangle ABC$ touches AB at K and AC at L, and J is the incenter of triangle BCD. Prove that KL bisects the segment AJ.

2. Triangle $\triangle GRB$ is dissected into 25 „small“ triangles as shown. All vertices of these triangles are painted in three colors so that the following conditions are satisfied:

Vertex G is painted in green, vertex R in red, and B in blue;

Each vertex on side GR is either green or red, each vertex on RB is either red or blue, and each vertex on GB is either green or blue. The vertices inside the big triangle are arbitrarily colored.



Prove that, regardless of the way of coloring, at least one of the 25 small triangles has vertices of three different colors.

3. Determine all pairs of natural numbers (x, n) that satisfy the equation

$$x^3 + 2x + 1 = 2^n.$$

Time allowed: 270 minutes.
Each problem is worth 7 points.

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Second Day

Belgrade , April 3, 2007

4. Let k be a natural number. For each function $f : \mathbb{N} \to \mathbb{N}$ define the sequence of functions $(f_m)_{m \geqslant 1}$ by $f_1 = f$ and $f_{m+1} = f \circ f_m$ for $m \geqslant 1$. Function f is called *k-nice* if for each $n \in \mathbb{N}$

$$f_k(n) = f(n)^k.$$

(a) For which k does there exist an injective k-nice function f?

(b) For which k does there exist a surjective k-nice function f?

5. In a scalene triangle ABC, AD, BE, CF are the angle bisectors ($D \in BC$, $E \in AC$, $F \in AB$). Points K_a, K_b, K_c on the incircle of triangle ABC are such that DK_a, EK_b, FK_c are tangent to the incircle and $K_a \notin BC$, $K_b \notin AC$, $K_c \notin AB$. Let A_1, B_1, C_1 be the midpoints of sides BC, CA, AB, respectively. Prove that the lines A_1K_a, B_1K_b, C_1K_c intersect on the incircle of triangle ABC.

6. Let k be a given natural number. Prove that for any positive numbers x, y, z with the sum 1 the following inequality holds:

$$\frac{x^{k+2}}{x^{k+1} + y^k + z^k} + \frac{y^{k+2}}{y^{k+1} + z^k + x^k} + \frac{z^{k+2}}{z^{k+1} + x^k + y^k} \geqslant \frac{1}{7}\,.$$

When does equality occur?

Time allowed: 270 minutes.
Each problem is worth 7 points.

SOLUTIONS

1. Let M be the point on AC such that $JM \parallel KL$. It is enough to prove that $AM = 2AL$.

From $\sphericalangle BDA = \alpha$ we obtain that $\sphericalangle JDM = 90^\circ - \frac{\alpha}{2} = \sphericalangle KLA = \sphericalangle JMD$; hence $JM = JD$ and the tangency point of the incircle of $\triangle BCD$ with CD is the midpoint T of segment MD. Therefore, $DM = 2DT = BD + CD - BC = AB - BC + CD$, which gives us

$$AM = AD + DM = AC + AB - BC = 2AL.$$

2. Consider those sides of small triangles that have a red endpoint and a blue endpoint. We call such sides *red-blue*.

Each red-blue side lying in the interior of triangle $\triangle ZCP$ is a side of exactly two small triangles. Moreover, each red-blue side lying on the boundary of $\triangle ZCP$ must lie on segment CP by the condition of the problem. Since point C is red and P blue, the number of red-blue sides on segment CP is odd.

It follows that there exists a small triangle having an odd number of red-blue sides. This triangle clearly has vertices of three different colors.

3. It is directly verified that the only solution with $n \leqslant 2$ is $(1, 2)$. We claim that there are no solution for $n \geqslant 3$.

Number x must be odd, so $x^2+2 \equiv 3 \pmod 8$. Now it follows from $x(x^2+2) \equiv -1 \pmod 8$ that $x \equiv 5 \pmod 8$. Moreover, since $3 \mid x(x^2+2)$ (if $3 \nmid x$ then $3 \mid x^2+2$), we obtain $2^n \equiv 1 \pmod 3$, which implies that n is even.

Adding 2 to both sides of the equation yields

$$(x+1)(x^2-x+3) = 2^n + 2.$$

Since n is even, 2^n is a perfect square and therefore -2 is a quadratic residue modulo each odd prime divisor p of $(x+1)(x^2-x+3)$. Hence

$$1 = \left(\frac{-2}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{2}{p}\right) = (-1)^{\frac{p-1}{2}}(-1)^{\frac{p^2-1}{8}} = (-1)^{\frac{(p-1)(p+5)}{8}},$$

from which we deduce that p is of the form $8k+1$ or $8k+3$. Being a product of such primes, the number x^2-x+3 itself must be of the same form. However, since $x \equiv 5 \pmod 8$, we have $x^2 - x + 3 \equiv 7 \pmod 8$ which is a contradiction.

Thus, the only solution of the given equation is $(x, n) = (1, 2)$.

4. Every function is 1-nice, and both answers for $k = 1$ are *yes*. From now on, we assume that $k \geqslant 2$. Every k-nice function is injective, as $f(m) = f(n)$ implies $m^k = f_k(m) = f_k(n) = n^k$ and hence $m = n$.

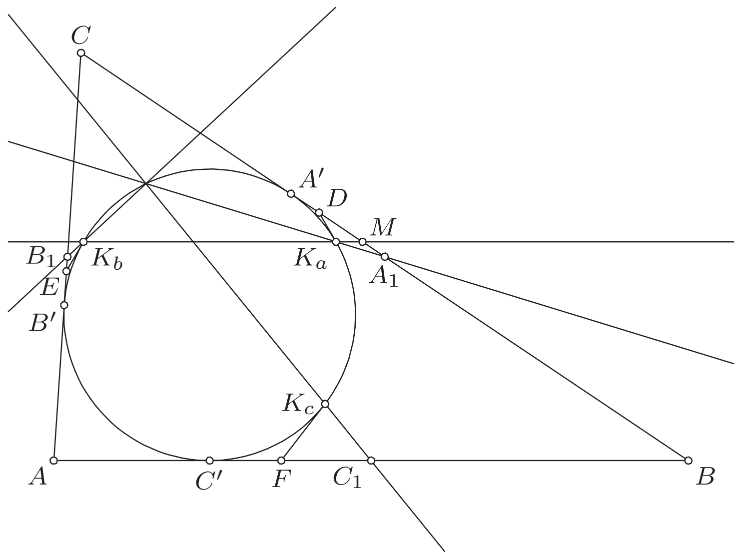
(a) Answer: *Yes*. We construct function f inductively as follows. Suppose that n is the smallest natural number at which the value of f has not been defined.

(1) If $n = 1$, then $f(n) = 1$;
(2) If $n = a^k$ for some integer $a > 1$, define $f(n) = f(a)^k$;
(3) If n is not a k-th power, take the smallest $k-1$ natural numbers $n_1, n_2, \dots, n_{k-1}$ that are not k-th powers and at which we have not yet defined values of f, and define $f(n_1) = n_2,\ f(n_2) = n_3,\ \dots,\ f(n_{k-1}) = n_1^k$.

The obtained function f is well-defined. We claim that it is k-nice. For each $n \in \mathbb{N}$ which is not a k-th power there are numbers $n_1, \dots, n_{k-1}$ from condition (3) such that $n_i = n$ for some $1 \leqslant i \leqslant k-1$. Then $f_k(n_i) = f_i(n_1^k) = f_i(n_1)^k = f(n_i)^k$. Similarly, if n is a k-th power, we have $n = n_i^{k^s}$ for some i and s, so by (2) it holds that $f_k(n) = f_k(n_i)^{k^s} = n_i^{k^{s+1}} = n^k$, justifying our claim.

(b) Answer: *No.* If f was surjective and k-nice, each a_0 would induce a sequence of positive integers $a_1, a_2, \dots$ such that $f(a_{k+1}) = a_k$ for all k, implying that $a_k^k = f_k(a_k) = a_0$, which is impossible unless a_0 is a k-th power.

5. We prove that the triangles $K_aK_bK_c$ and $A_1B_1C_1$ are homothetic. To prove this, it is enough to show that $K_aK_b \parallel A_1B_1$, i.e. $K_aK_b \parallel AB$ (the same will analogously follow for the other pairs of sides).
Let $M = K_aK_b \cap BC$, S be the incenter and T be an arbitrary point on the incircle. Denote $\alpha = \sphericalangle BAS$, $\beta = \sphericalangle CBS$, $\gamma = \sphericalangle ACS$. Using oriented angles modulo 180° we obtain $\sphericalangle B'EB = \beta + 2\gamma$, and analogously $\sphericalangle A'DA = \alpha + 2\beta$; therefore $\sphericalangle A'DK_a = 2\alpha + 4\beta$. Furthermore, $\sphericalangle B'TK_b = \sphericalangle B'SE = 90^\circ + \sphericalangle B'ES = \gamma - \alpha$ and analogously $\sphericalangle A'TK_a = \beta - \gamma$. Also, $\sphericalangle A'TB' = \sphericalangle A'SC = 90^\circ + \sphericalangle A'CS = \alpha + \beta$. Finally, $\sphericalangle K_aTK_b = \sphericalangle K_aTA' + \sphericalangle A'TB' + \sphericalangle B'TK_b = 2\gamma$.
From triangle DK_aM we obtain $\sphericalangle CMK_a = \sphericalangle CDK_a + \sphericalangle DK_aM = \sphericalangle A'DK_a + \sphericalangle DK_aK_b = (2\alpha + 4\beta) + \sphericalangle K_aTK_b = (2\alpha + 4\beta) + 2\gamma = 2\beta$. Hence, $\sphericalangle CMK_a = \sphericalangle CBA$, which implies that $K_aK_b \parallel AB$, as claimed. Thus, triangles $K_aK_bK_c$ and $A_1B_1C_1$ are homothetic.
Note that the coefficient of the homothety is positive: otherwise the segments K_aA_1, K_bB_1, K_cC_1 would concur. However, if $\alpha > \beta$ then points C_1 and K_c and consequently the entire segment K_cC_1 lie within the quadrilateral $SFBD$. Thus, assuming w.l.o.g. that $\alpha > \beta > \gamma$, we have $K_cC_1 \subset SFBD$, but $K_aA_1 \subset SDCE$, which means that these two segments are disjoint.
Since the triangles $K_aK_bK_c$ and $A_1B_1C_1$ are homothetic, so are their circumcircles which are the Euler circle and incircle of triangle ABC, respectively, whereby it is known that these two circles are internally tangent at the Feuerbach point of $\triangle ABC$. This (together with the positivity of the coefficient of homothety) implies that the center of homothety if precisely the Feuerbach point. Therefore A_1K_a, B_1K_b, C_1K_c meet in the Feuerbach point which lies on the incircle of $\triangle ABC$.



Alternative Solution. Let the incircle of $\triangle ABC$ be the unit circle in the complex plane. Then $a = \frac{2b'c'}{b'+c'}$, $b = \frac{2a'c'}{a'+c'}$, $c = \frac{2a'b'}{a'+b'}$. Therefore

$$a_1 = \frac{b+c}{2} = \frac{a'^2b' + a'^2c' + 2a'b'c'}{(a'+b')(a'+c')}.$$

The number k_a is determined from the condition $\frac{k_a}{a} = \overline{\left(\frac{a'}{a}\right)}$, giving us $k_a = \frac{1}{a'}\frac{a}{\overline{a}} = \frac{b'c'}{a'}$. Next we find the second intersection z of the incircle (so $|z| = 1$) and the line K_aA_1 (so $\frac{z-k_a}{a_1-k_a} = \overline{\left(\frac{z-k_a}{a_1-k_a}\right)}$). The second equation can be transformed into

$$\overline{(a_1 - k_a)}(z - k_a) = (\frac{1}{z} - \frac{1}{k_a})(a_1 - k_a),$$

which (since $z \neq k_a$) implies $\overline{(a_1 - k_a)} = -\frac{1}{zk_a}(a_1 - k_a)$, so

$$z = -\frac{1}{k_a}\frac{a_1 - k_a}{\overline{(a_1 - k_a)}} = -\frac{(a'^2 - b'c')(a'b' + a'c' + b'c')}{(b'c' - a'^2)(a' + b' + c')} = \frac{a'b' + a'c' + b'c'}{a' + b' + c'}.$$

Since this expression is symmetric in a', b', c', we analogously obtain that the lines K_bB_1 and K_cC_1 meet the incircle in the same point, which proves our statement.

6. The left hand side of the inequality is symmetric, so we can assume w.l.o.g. that $x \geqslant y \geqslant z$. Then

$$x^{k+1} + y^k + z^k \leqslant y^{k+1} + z^k + x^k \leqslant z^{k+1} + x^k + y^k.$$

Indeed, it suffices to show the first inequality, i.e. that $x^{k+1} + y^k \leqslant y^{k+1} + x^k$. This inequality is equivalent to $\left(\frac{y}{x}\right)^k \leqslant \frac{1-x}{1-y}$. Since $y \leqslant x$, it is enough to show that

$\frac{y}{x} \leqslant \frac{1-x}{1-y}$, which is equivalent to the inequality $0 \leqslant x - x^2 - y + y^2 = (x-y)(1-x-y) = (x-y)z$ which is true. Thus, applying the Chebyshev inequality on the triples $(x^{k+2}, y^{k+2}, z^{k+2})$ and $\left(\frac{1}{x^{k+1}+y^k+z^k}, \frac{1}{y^{k+1}+z^k+x^k}, \frac{1}{z^{k+1}+x^k+y^k}\right)$ yields

$$\sum_{\text{cyc}} \frac{x^{k+2}}{x^{k+1}+y^k+z^k} \geqslant \frac{1}{3} \sum_{\text{cyc}} x^{k+2} \sum_{\text{cyc}} \frac{1}{x^{k+1}+y^k+z^k} = L.$$

If we apply the Chebyshev inequality once again on the triples (x, y, z) and $(x^{k+1}, y^{k+1}, z^{k+1})$ in L we obtain

$$L \geqslant \frac{1}{3} \cdot \frac{1}{3} \sum_{\text{cyc}} x \sum_{\text{cyc}} x^{k+1} \sum_{\text{cyc}} \frac{1}{x^{k+1}+y^k+z^k} = L'. \qquad (*)$$

The Cauchy-Schwartz inequality gives us

$$\sum_{\text{cyc}} \frac{1}{x^{k+1}+y^k+z^k} \sum_{\text{cyc}} (x^{k+1}+y^k+z^k) \geqslant 9,$$

so

$$L' \geqslant \frac{x^{k+1}+y^{k+1}+z^{k+1}}{x^{k+1}+y^{k+1}+z^{k+1}+2(x^k+y^k+z^k)}.$$

Thus we are done if we prove that

$$3(x^{k+1}+y^{k+1}+z^{k+1}) \geqslant x^k+y^k+z^k.$$

The last inequality follows from the Chebyshev inequality applied on (x, y, z) and (x^k, y^k, z^k).

Equality in all the applied inequalities holds if and only if $x = y = z$, i.e. if and only if $x = y = z = \frac{1}{3}$.