23-rd Balkan Mathematical Olympiad

Agros, Cyprus – April 29, 2006

1. If a, b, c are positive numbers, prove the inequality

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{1+abc}\,.$$

2. A line m intersects the sides AB, AC and the extension of BC beyond C of the triangle ABC at points D, F, E, respectively. The lines through points A, B, C which are parallel to m meet the circumcircle of triangle ABC again at points A_1, B_1, C_1, respectively. Show that the lines A_1E, B_1F, C_1D are concurrent.

3. Determine all triples (m, n, p) of positive rational numbers such that the numbers

$$m + \frac{1}{np}, \quad n + \frac{1}{pm}, \quad p + \frac{1}{mn}$$

are integers.

4. Given a positive integer m, consider the sequence (a_n) of positive integers defined by the initial term $a_0 = a$ and the recurrent relation

$$a_{n+1} = \begin{cases} \dfrac{a_n}{2} & \text{if } a_n \text{ is even,} \\ a_n + m & \text{if } a_n \text{ is odd.} \end{cases}$$

Find all values of a for which this sequence is periodic (i.e. there exists $d > 0$ such that $a_{n+d} = a_n$ for all n).