Serbia and Montenegro Team Selection Tests 2005

Selection Test for Balkan MO

Budva, April 17

1. A sequence is defined by $x_1 = 1$, $x_2 = 4$ and $x_{n+2} = 4x_{n+1} - x_n$ for $n \geq 1$. Find all natural numbers m such that the number $3x_n^2 + m$ is a perfect square for all natural numbers n.

2. Determine the number of 100-digit numbers whose all digits are odd, and in which every two consecutive digits differ by 2.

3. (a) Show that there exists a multiple of 2005 whose sum of (decimal) digits equals 2.

   (b) Let x_n denote the number obtained by writing natural numbers from 1 to n one after another (for example, $x_1 = 1$, $x_2 = 12, \dots, x_{13} = 12345678910111213$). Prove that the sequence $x_1, x_2, \dots$ contains infinitely many terms that are divisible by 2005.

First Test for IMO

Belgrade, May 31

1. Prove that there is no rational number r such that $\cos r\pi = \dfrac{3}{5}$.

2. A convex angle xOy and a point M inside it are given in the plane. Prove that there is a unique point P in this plane with the following property:
   - For any line l through M, meeting the rays x or y (or their extensions) at X and Y, the angle $\angle XPY$ is not obtuse.

3. Find all polynomials $P(x)$ that satisfy $P(x^2 + 1) = P(x)^2 + 1$.

Second Test for IMO

Belgrade, June 1

1. Let T be the centroid of a triangle ABC. Prove that

$$\frac{1}{\sin \angle TAC} + \frac{1}{\sin \angle TBC} \geq 4.$$

2. If a, b, c are positive numbers with $abc = 1$, prove that

$$\frac{a}{a^2+2} + \frac{b}{b^2+2} + \frac{c}{c^2+2} \leq 1.$$

3. We say that n squares in an $n \times n$ board are *scattered* if no two of them are in the same row or column. In every square of this board is written a natural number so that the sum of numbers in n scattered squares is always the same, and no row or column contains two equal numbers. It turned out that the numbers on the main diagonal are arranged in the increasing order, and that their product is the smallest among all products of n scattered numbers. Prove that the scattered numbers with the greatest product are exactly those on the other diagonal.