40-th International Mathematical Olympiad

Bucharest, Romania, July 10–22, 1999

First Day – July 16

1. A set S of points in the plane will be called *completely symmetric* if it has at least three elements and satisfies the following condition: For every two distinct points A, B from S the perpendicular bisector of the segment AB is an axis of symmetry for S.

   Prove that if a completely symmetric set is finite, then it consists of the vertices of a regular polygon. *(Estonia)*

2. Let $n \geq 2$ be a fixed integer. Find the least constant C such that the inequality
$$\sum_{i<j} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_i x_i \right)^4$$
holds for every $x_1, \ldots, x_n \geq 0$ (the sum on the left consists of $\binom{n}{2}$ summands). For this constant C, characterize the instances of equality. *(Poland)*

3. Let n be an even positive integer. We say that two different cells of an $n \times n$ board are *neighboring* if they have a common side. Find the minimal number of cells on the $n \times n$ board that must be marked so that every cell (marked or not marked) has a marked neighboring cell. *(Belarus)*

Second Day – July 17

4. Find all pairs of positive integers (x, p) such that p is a prime, $x \leq 2p$, and x^{p-1} is a divisor of $(p-1)^x + 1$. *(Taiwan)*

5. Two circles Ω_1 and Ω_2 touch internally the circle Ω in M and N, and the center of Ω_2 is on Ω_1. The common chord of the circles Ω_1 and Ω_2 intersects Ω in A and B. MA and MB intersect Ω_1 in C and D. Prove that Ω_2 is tangent to CD. *(Russia)*

6. Find all the functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy
$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$
for all $x, y \in \mathbb{R}$. *(Japan)*