39-th International Mathematical Olympiad

Taipei, Taiwan, July 10–21, 1998

First Day – July 15

1. A convex quadrilateral $ABCD$ has perpendicular diagonals. The perpendicular bisectors of AB and CD meet at a unique point P inside $ABCD$. Prove that $ABCD$ is cyclic if and only if triangles ABP and CDP have equal areas. *(Luxembourg)*

2. In a contest, there are m candidates and n judges, where $n \geq 3$ is an odd integer. Each candidate is evaluated by each judge as either pass or fail. Suppose that each pair of judges agrees on at most k candidates. Prove that
$$\frac{k}{m} \geq \frac{n-1}{2n}.$$
(India)

3. For any positive integer n, let $\tau(n)$ denote the number of its positive divisors (including 1 and itself). Determine all positive integers m for which there exists a positive integer n such that $\dfrac{\tau(n^2)}{\tau(n)} = m$. *(Belarus)*

Second Day – July 16

4. Determine all pairs (x, y) of positive integers such that $x^2y + x + y$ is divisible by $xy^2 + y + 7$. *(Great Britain)*

5. Let I be the incenter of triangle ABC. Let K, L, and M be the points of tangency of the incircle of ABC with AB, BC, and CA, respectively. The line t passes through B and is parallel to KL. The lines MK and ML intersect t at the points R and S. Prove that $\angle RIS$ is acute. *(Ukraine)*

6. Determine the least possible value of $f(1998)$, where f is a function from the set $\mathbb{N}$ of positive integers into itself such that for all $m, n \in \mathbb{N}$,
$$f(n^2 f(m)) = m[f(n)]^2.$$
(Bulgaria)