Serbian Mathematical Olympiad 2011

Belgrade, April 2–3

First Day

1. Let $n \geq 2$ be a natural number and suppose that positive numbers a_0, a_1, $\ldots$, a_n satisfy the equality

$$(a_{k-1} + a_k)(a_k + a_{k+1}) = a_{k-1} - a_{k+1} \text{ for each } k = 1, 2, \ldots, n-1.$$

Prove that $a_n < \frac{1}{n-1}$.

2. Let N be an odd positive integer such that the numbers $\varphi(n)$ and $\varphi(n+1)$ are both powers of two ($\varphi(n)$ denotes the number of natural numbers cop rime to n and not exceeding n). Prove that $n+1$ is a power of two for $n = 5$.

3. Let H be the orthocenter and O be the circumcenter of an acute-angled triangle ABC. Points D and E are the feet of the altitudes from A and B, respectively. Lines OD and BE meet at point K, and lines OE and AD meet at point L. Let X be the second intersection point of the circumcircles of triangles HKD and HLE, and let M be the midpoint of side AB. Prove that points K, L, and M are collinear if and only if X is the circumcenter of $\triangle EOD$.

Second Day

4. Points M, X, and Y are taken on sides AB, AC, and BC respectively of a triangle ABC such that $AX = MX$ and $BY = MY$. Let K and L be the midpoints of segments AY and BX respectively, and let O be the circumcenter of $\triangle ABC$. If points O_1 and O_2 are symmetric to point O with respect to K and L, respectively, show that the points X, Y, O_1, and O_2 lie on a circle.

5. Do there exist integers a, b, and c greater than 2011 such that in the decimal system they satisfy

$$\left(a + \sqrt{b}\right)^c = \ldots 2010{,}2011\ldots?$$

(The symbol , represents the decimal point.)

6. Set T consists of 66 points, and set P consists of 16 lines in the plane. We say that a point $A \in T$ and a line $l \in P$ form an *incident pair* if $A \in l$. SHow that the number of incident pairs cannot exceed 159, and that there is such a configuration with exactly 159 incident pairs.

Time allowed: 270 minutes.
Each problem is worth 7 points.

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