Serbian Mathematical Olympiad 2010

Niš, April 6–7

First Day

1. Some of n towns are connected by two-way airlines. There are m airlines in total. For $i = 1, 2, \dots, n$, let d_i be the number of airlines going from town i. If $1 \le d_i \le 2010$ for each $i = 1, 2, \dots, 2010$, prove that $\sum_{i=1}^{n} d_i^2 \le 4022m - 2010n$.

2. In an acute-angled triangle ABC, M is the midpoint of side BC, and D, E, and F the feet of the altitudes from A, B, and C, respectively. Let H be the orthocenter of $\triangle ABC$, S the midpoint of AH, and G the intersection of FE and AH. If N is the intersection of the median AM and the circumcircle of $\triangle BCH$, prove that $\angle HMA = \angle GNS$.

3. Let A be an infinite set of positive integers. Find all natural numbers n such that fro each $a \in A$:
$$a^n + a^{n-1} + \cdots + a^1 + 1 \mid a^{n!} + a^{(n-1)!} + \cdots + a^{1!} + 1.$$

Second Day

4. Let O be the circumcenter of triangle ABC. A line through O intersects the sides CA and CB at points D and E respectively, and meets the circumcircle again at point $P \ne O$ inside the triangle. A point Q on side AB is such that $\frac{AQ}{QB} = \frac{DP}{PE}$. Prove that $\angle APQ = 2\angle CAP$.

5. An $n \times n$ table whose cells are numerated with numbers $1, 2, \dots, n^2$ in some order is called *Naissus* if all products of n numbers written in n *scattered* cells give the same residue when divided by $n^2 + 1$. Does there exist a Naissus table for:

   (a) $n = 8$;

   (b) $n = 10$?

   (n cells are *scattered* if no two are in the same row or column.)

6. Let a_0 and a_n be different divisors of a natural number m, and $a_0, a_1, a_2, \dots, a_n$ be a sequence of natural numbers such that it satisfies
$$a_{i+1} = |a_i \pm a_{i-1}|, \text{ for } 0 < i < n.$$
If gcd $(a_0, \dots, a_n) = 1$, show that there is a term of the sequence that is smaller than $\sqrt{m}$.

Time allowed: 270 minutes.
Each problem is worth 7 points.

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