Bulgarian Mathematical Olympiad 1992, IV Round

First Day

1. Through a random point C_1 from the edge DC of the regular tetrahedron $ABCD$ is drawn a plane, parallel to the plane ABC. The plane constructed intersects the edges DA and DB at the points A_1, B_1 respectively. Let the point H is the midlepoint of the height through the vertex D of the tetrahedron $DA_1B_1C_1$ and M is the center of gravity (medicenter) of the triangle ABC_1. Prove that the dimension of the angle HMC doesn't depend of the position of the point C_1. (Ivan Tonov)

2. Prove that there exists 1904-element subset of the set $\{1, 2, \dots, 1992\}$, which doesn't contain an arithmetic progression consisting of 41 terms.

   (Ivan Tonov)

3. Let m and n are fixed natural numbers and Oxy is a coordinate system in the plane. Find the total count of all possible situations of $n+m-1$ points $P_1(x_1, y_1)$, $P_2(x_2, y_2), \dots, P_{n+m-1}(x_{n+m-1}, y_{n+m-1})$ in the plane for which the following conditions are satisfied:

   (i) The numbers x_i and y_i ($i = 1, 2, \dots, n+m-1$) are integer (whole numbers) and $1 \le x_i \le n$, $1 \le y_i \le m$.

   (ii) Every one of the numbers $1, 2, \dots, n$ can be found in the sequence $x_1, x_2, \dots, x_{n+m-1}$ and every one of the numbers $1, 2, \dots, m$ can be found in the sequence $y_1, y_2, \dots, y_{n+m-1}$.

   (iii) For every $i = 1, 2, \dots, n+m-2$ the line P_iP_{i+1} is parallel to one of the coordinate axes.

   (Ivan Gochev, Hristo Minchev)

Second day

4. Let p is a prime number in the form $p = 4k+3$. Prove that if the numbers x_0, y_0, z_0, t_0 are solution of the equation: $x^{2p} + y^{2p} + z^{2p} = t^{2p}$, then at least one of them is divisible by p. (Plamen Koshlukov)

5. Points D, E, F are middlepoints of the sides AB, BC, CA of the triangle ABC. Angle bisectors of the angles BDC and ADC intersects the lines BC and AC respectively at the points M and N and the line MN intersects the line CD at the point O. Let the lines EO and FO intersects respectively the lines AC and BC at the points P and Q. Prove that $CD = PQ$. (Plamen Koshlukov)

6. There are given one black box and n white boxes (n is a random natural number). White boxes are numbered with the numbers $1, 2, \dots, n$. In them are put n balls. It is allowed the following rearrangement of the balls: if in the box with number

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k there are exactly k balls that box is made empty - one of the balls is put in the black box and the other $k-1$ balls are put in the boxes with numbers: $1, 2, \ldots, k-1$. (Ivan Tonov)

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