Bulgarian Mathematical Olympiad 1994, III Round

First Day, 23 april 1994

1. Let $n > 1$ is a natural number and $A_n = \{x \in \mathbb{N} \mid (x,n) \neq 1\}$ is the set of all natural numbers that aren't mutually prime (coprime) with n. We say that the n is *interesting* if for every two numbers $x, y \in A_n$ it is true $x + y \in A_n$.

   (a) Prove that the number 43 is *interesting*.

   (b) Prove that 1994 isn't *interesting*.

   (c) Find all *interesting* numbers.

2. Around some circle are circumscribed a square and a triangle. Prove that at least a half from the square's perimeter lies inside the triangle.

3. Around unlimited chessboard is situated the figure (p,q)-*horse* which on each its move moves p-fields horizontally or vertically and q-fields on direction perpendicular to the previous direction (the ordinary chess horse is $(2,1)$-*horse*). Find all pairs of natural numbers (p,q) for which the (p,q)-*horse* can reach to all possible fields on the chessboard with limited count of moves.

Second day, 24 april 1994

4. The sequence $a_0, a_1, \ldots, a_n$ satisfies the condition:

$$a_{n+1} = 2^n - 3a_n \qquad , \qquad n = 0, 1,$$

   (a) Express the common term a_n as a function of a_0 and n.

   (b) Find a_0 if $a_{n+1} > a_n$ for each natural number n.

5. It is given a rectangular parallelepiped $ABCDA_1B_1C_1D_1$. The perpendiculars from the point A to the lines A_1B, A_1C and A_1D intersects the lines A_1B_1, A_1C and A_1D intersects the lines A_1B_1, A_1C_1 and A_1D_1 at the points M, N and P respectively.

   (a) Prove that M, N and P lies on a common line.

   (b) If E and F are the feets of the perpendiculars from A to A_1B and A_1D. Prove that the lines PE, MF and AN have a single point in common.

6. Let a, b and c are real numbers such that equation $ax^2 + bx + c = 0$ have real roots. Prove that if $|a(b-c)| > \left|b^2 - ac\right| + \left|c^2 - ab\right|$ then the equation have at least one root from the interval $(0,2)$.

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