Dutch Mathematical Olympiad 1999

Second Round

1. Let $f:\mathbb{Z}\to\{-1,1\}$ be a function such that

$$f(mn)=f(m)f(n),\quad \text{for all } m,n\in\mathbb{Z}.$$

Prove that there exists a positive integer $a\in[1,12]$ such that $f(a)=f(a+1)=1$.

2. Some of the cells of the 9×9 board are painted black and the others are painted in white. Determine the total number of black squares if each 2×3 and each 3×2 rectangle contains exactly 2 black squares.

3. Let M be the center of the square $ABCD$ whose diagonal has length 2. l is a line whose distance d from M is greater than 1. Denote by A', B', C', D' the projections of A, B, C, D to l. Prove that $AA'^2+BB'^2+CC'^2+DD'^2$ depends only on d.

4. All entries of a 8×8 matrix are positive integers. One may repeatedly transform the entries of the matrix according to the following rules:

   (i) Multiply all entries in some row by 2.

   (ii) Subtract 1 from all entries in some column.

   Prove that it is possible to transform the given matrix into the zero matrix.

5. Given a non-negative integer c, define $a_n=n^2+c$ for $n\geq 1$. Let $d_n=\gcd(a_n,a_{n+1})$.

   (a) If $c=0$, prove that $d_n=1$ for all n.

   (b) If $c=1$ prove that $d_n\in\{1,5\}$ for all $n\geq 1$.

   (c) Prove that $d_n\leq 4c+1$ for all $n\geq 1$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com