Serbian Mathematical Olympiad 2009

Belgrade, April 12–13

First Day

1. Given a scalene triangle ABC, denote $\alpha = \angle BAC$ and $\beta = \angle ABC$. Assume that bisectors of these angles intersect the opposite sides at D and E respectively. Prove that the acute angle between the lines DE and AB is not greater than $\frac{|\alpha-\beta|}{3}$.

2. Find the smallest positive integer that is divisible by 2009 and whose sum of digits is 2009.

3. Find the biggest positive integer n for which there exist different sets $S_1, \ldots, S_n$ such that:

   1° $|S_i \cup S_j| \leq 2004$ for every two $1 \leq i, j \leq n$ and

   2° $S_i \sup S_j \cup S_k = \{1, 2, \ldots, 2008\}$ for every triple (i, j, k) of integers such that $1 \leq i < j < k \leq n$.

Second Day

4. Let $n \in \mathbb{N}$ and A_n set of all permutations $(a_1, \ldots, a_n)$ of the set $\{1, 2, \ldots, n\}$ for which
$$k | 2(a_1 + \cdots + a_k), \text{ for all } 1 \leq k \leq n.$$
   Find the number of elements of the set A_n.

5. Let x, y, z be positive real numbers such that $xy + yz + zx = x + y + z$. Prove the inequality
$$\frac{1}{x^2+y+1} + \frac{1}{y^2+z+1} + \frac{1}{z^2+x+1} \leq 1.$$

6. Let k be a circle inscribed in a triangle ABC whose center is S. Circle k touches the sides BC, CA, and AB at P, Q, R, respectively. Line QR intersects line BC at M. A circle that passes through B and C touches k at N. The circumcircle of $\triangle MNP$ intersects AP at L, different than P. Prove that the points S, L, and M belong to a line.

Time allowed: 270 minutes.
Each problem is worth 7 points.

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