Moldovan Team Selection Tests 2007

First Test

1. Let M, N, P be the midpoints of the sides BC, CA, and AB, respectively, of the triangle ABC. The lines AM, BN, and CP intersect the circumcircle of $\triangle ABC$ again at A_1, B_1, C_1, respectively. Prove that $S_{ABC} \leq S_{BCA_1} + S_{CAB_1} + S_{ABC_1}$.

2. Let p be a prime number. Consider p consecutive integers m_1, m_2, $\dots$, m_p. Choose a permutation σ of 1, 2, $\dots$, p. Show that there exist two different numbers $k, l \in \{1, 2, \dots, p\}$ such that $lm_k m_{\sigma(k)} - m_l m_{\sigma(l)}$ is divisible by p.

3. Let ABC be a triangle whose all angles are smaller than or equal to 120°. Let F be the point in the interior of $\triangle ABC$ such that $\angle AFB = \angle BFC = \angle CFA = 120^\circ$. For each one of the triangles BFC, CFA, and AFB draw the line connecting its circumcenter with its centroid. Prove that these three lines pass through a common point.

4. Consider a convex polygon $A_1A_2 \dots A_n$ and a point M inside it. The lines A_iM intersect the perimeter of the polygon second time in the points B_i. The polygon is called balanced if all sides of the polygon contain exactly one of the points B_i (strictly in the interior). Find all balanced polygons.

Second Test

1. Find the smallest positive integers m and k such that:

   (a) There exist $2m+1$ consecutive natural numbers whose sum of cubes is also a cube.

   (b) There exist $2k+1$ consecutive natural numbers whose sum of squares is also a square.

2. If I is the incenter of a triangle ABC and R is the radius of its circumcircle prove that
$$AI + BI + CI \leq 3R.$$

3. Let M, N be the points inside the angle BAC such that $\angle MAB = \angle NAC$. If M_1, M_2, and N_1, N_2 are projections of M and N on AB, AC, respectively prove that M, N, P, and the intersectiohn of M_1N_2 with N_1M_2 are collinear.

4. Five points are given in the plane such that no three of them are collinear. The convex hull of this points has area S. Prove that there exist three among those five points that form a triangle of an area at most $\frac{5-\sqrt{5}}{10}S$.

Third Test

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com

1. Let $a_1, a_2, \dots, a_n \in [0,1]$ and let $S = a_1^3 + \cdots + a_n^3$. Prove that
$$\frac{a_1}{2n+1+S-a_1^3} + \frac{a_2}{2n+1+S-a_2^3} + \cdots + \frac{a_n}{2n+1+S-a_n^3} \leq \frac{1}{3}.$$

2. Find all polynomials f with integer coefficients such that $f(p)$ is prime for all prime numbers p.

3. Consider a triangle ABC with corresponding sides a, b, c, inradius r and circumradius R. If r_A, r_B, r_C are the radii of the respective excircles of the triangle show that
$$a^2\left(\frac{2}{r_A} - \frac{r}{r_B r_C}\right) + b^2\left(\frac{2}{r_B} - \frac{r}{r_C r_A}\right) + c^2\left(\frac{2}{r_C} - \frac{r}{r_A r_B}\right) = 5(R+3r).$$

4. Given n distinct points in a plane, consider the number $\tau(n)$ of pairs of these points whose distance is exactly 1. Show that $\tau(n) \leq \frac{n^2}{3}$.

Fourth Test

1. Show that a plane cannot be represented as the union of the inner regions of a finite number of parabolas.

2. If b_1, b_2, $\dots$, b_n are non-negative real numbers not all equal to 0, prove that the polynomial $x^n - b_1x^{n-1} - b_2x^{n-2} - \cdots - b_n$ has only one positive root p, which is simple. Prove that the absolute value of all roots of the polynomial are smaller than or equal to p.

3. A circle is tangent to the sides AB, AC, and to the circumcircle of $\triangle ABC$ at points P, Q, and R, respectively. Let S be the point where AR meets PQ. Prove that $\angle SBA = \angle SCA$.

4. SHow that there are infinitely many prime numbers p with the following property: There exists a natural number n, not dividing $p-1$, such that $p \mid n!+1$.