Bulgarian Mathematical Olympiad 2007
Regional Round, April 14-15

Grade 9

First Day

1. Find the real solutions of the equation:

$$\sqrt{x-3}+\sqrt{7-x}=x^2-10x+23$$

2. The quadrilateral $ABCD$ that have no two parallel sides is inscribed in a circle with radii 1. The point E belongs to AB and F belongs to CD are such that $CE \| AD$ and $CF \| AB$. It is also known that the circumscribed circle around the triangle CDF intersects for second time the diagonal AC in an internal point. Find the biggest possible value of the expression: $AB \cdot AE + AD \cdot AF$.

3. Find the smallest possible value of the expression:

$$M = x+\frac{y^2}{9x}+\frac{3z^2}{32y}+\frac{2}{z}$$

Second day

4. Prove that there doesn't exist values for a real paramether a for which the system:

$$\begin{cases} x^2 = x+ay+1 \\ y^2 = ax+y+1 \end{cases}$$

have exactly three different solutions.

5. Find all even natural numbers n and all real numbers a, for which the remainder of division of the polynomial $x^n - x^{n-1} + ax^4 + 1$ by the polynomial $x^2 - a^2$ is equal to $97-(a+14)x$.

6. It is given a reqular 16-gon $A_1 \cdots A_{16}$ which vertices lie on a circle (circumference) with center O. Is is possible to be chosen some vertices ot the 16-gon in such a way when we rotate the 16-gon around O by angles: $\frac{360^\circ}{16}$, $2\cdot\frac{360^\circ}{16}$, $\cdots$, $16\cdot\frac{360^\circ}{16}$ the segments connecting the vertices chosen to all the sides and diagonals of the 16-gon exactly two times.

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Grade 10

First Day

1. Solve the equation inequality:

$$\frac{\sqrt{-x^2+6x+1}+5x-15}{2x-5} \geq 2$$

2. It is given a triangle ABC. D, E, F are the tangent points of externally inscribed circles to the sides BC, CA, AB respectively.

   (a) Prove that AD, BE and CF intersects at a common point.

   (b) Prove that if the common point of AD, BE and CF lies on the incircle of the triangle, then the perimeter of the triangle is four times greater than its smallest side.

3. The natural numbers $a_1, a_2, \ldots, a_n$, $n \geq 3$ are such that:

$$b_1 = \frac{a_n + a_2}{a_1} \quad , \quad b_2 = \frac{a_1 + a_2}{a_2} \quad , \quad \cdots , \quad b_n = \frac{a_{n-1} + a_1}{a_n}$$

   are integer numbers. Prove that $b_1 + b_2 + \cdots + b_n \leq 3n - 1$.

Second day

4. Find the values of the real parameter a, for which the equation

$$\log_{x-a}(x+a) = 2$$

   have only one solution.

5. Find the count of all pairs of natural numbers (m,n) which are solutions of the system:

$$\begin{cases} 47^m - 48^n + 1 \equiv 0 \pmod{61} \\ 3m + 2n = 1000 \end{cases}$$

   have exactly three different solutions.

6. It is given a reqular 16-gon $A_1 \cdots A_{16}$ which vertices lie on a circle (circumference) with center O. Is is possible to be chosen some vertices ot the 16-gon in such a way when we rotate the 16-gon around O by angles: $\frac{360^\circ}{16}$, $2 \cdot \frac{360^\circ}{16}$, $\cdots$, $16 \cdot \frac{360^\circ}{16}$ the segments connecting the vertices chosen to all the sides and diagonals of the 16-gon exactly two times.

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Grade 11

First Day

1. Find all real values of the parameter a for which the system:

$$\begin{cases} \sin x + \cos y = 4a + 6 \\ \cos x + \sin y = 3a + 2 \end{cases}$$

have a solution.

2. Let n is a natural number and $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ are positive numbers. Prove the inequality:

$$(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n) + 2^{n-1} \left(\frac{1}{a_1 b_1} + \frac{1}{a_2 b_2} + \cdots + \frac{1}{a_n b_n} \right) \geq 2^{n-1}(a+2)$$

When does equality holds?

3. For the set A composed from real numbers we denote with A^+ the count of different numbers can be achieved as a sum of two (not necessary different) numbers and A^- is the count of different positive numbers that can be achieved as a difference between two numbers from A. Let A is such 2007-element set for which A^+ have maximal possible value. Find A^-.

Second day

4. It is given the sequence $\{a_n\}_{n=1}^{\infty}$ defined with the equalities: $a_1 = 4$, $a_2 = 3$ and $2a_{n+1} = 3a_n - a_{n-1}$ for $n \geq 2$.

   (a) Prove that the sequence converges and find its limit.

   (b) Calculate the following limit:

$$\lim_{n\to\infty} \frac{(a_n - 1)(a_n - 2)}{\left(\sqrt{a_n + 2} - 2\right)\left(\sqrt{12a_n - 8} - 2\right)}$$

5. Let the point D is a point from the side AB of the triangle ABC. We denote with P and M respectively the incenters of the triangles ADC and BDC. Q and N are the centers of externally inscribed circles in ADC to the side AD and in BDC to the side BD respectively. Let K and L are the symmetric points of Q and N to the line AB.

   (a) Prove that the lines AB, QN and PM intersects at a common point or they are parallel.

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(b) Prove that if the points M, P, K and L lies on a common circle, then ABC is isosceles.

6. Let a and b are natural numbers and $a = 4k+3$ for some integer number k. Prove that if the equation:
$$x^2 + (a-1)y^2 + az^2 = b^n$$
have solutions in positive integers x, y, and z for $n = 1$ then the equation have positive solution for every natural number n.

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Grade 12

First Day

1. Find all real values of the parameter a for which there exists two mutually perpendicular tangents to the function graph:

$$f(x) = ax + \sin x.$$

2. In sphere with radii 2 is inscribed a pyramid $ABCD$ for which $AB = 2$, $CD = \sqrt{7}$ and $\angle ABC = \angle BAD = 90^\circ$. Find the angle between the lines AD and BC.

3. Prove that if x and y are integer numbers, then the number $x^2(y-2) + y^2(x-2)$ is not prime.

Second day

4. Let I is the center of the externally inscribed circle tangent to the side AB of the triangle ABC. S is the symmetric point of I with respect to AB. The line through S, perpendicular to BI, intersects the line AI at the point T. Prove that $CI = CT$.

5. Solve the equation:
$$\frac{8^x - 2^x}{6^x - 3^x} = 2$$

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) \le 0$ and $f(x+y) \le x + f(f(x))$ for any two numbers x, y from $\mathbb{R}$.

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