Dutch Mathematical Olympiad 2000

Second Round

1. For integers a and b we say that a is a *power* of b if there exists a positive integer n such that $a = b^n$. We say that a is a multiple of b if there exists an integer n such that $a = bn$. Let x, y, and z be positive integers such that z is a power of both x and y. For each of the following statements, determine (with proof) whether it is true or false:

   (a) The number $x + y$ is even.

   (b) One of x and y is the multiple of the other.

   (c) One of x and y is a power of the other.

   (d) There exists an integer v such that both x and y are powers of v.

   (e) For each power x_1 of x and each power y_1 of y, there exists an integer w that is a power of both x_1 and y_1.

   (f) There exists a positive integer k such that $x^k > y$.

2. The first box contains 600 identical red balls, the second box contains 600 identical white balls and the third box contains 600 identical blue balls. In how many ways one can choose 900 balls from these boxes?

3. Two similar isosceles triangles QPA and SPB are constructed outside the parallelogram $PQRS$ ($PQ = AQ$, $PS = BS$). Prove that $\triangle RAB \sim \triangle QPA \sim \triangle SPB$.

4. Each of fifteen boys holds a ball. No two distances between the boys are the same. Each boy throws the ball to the boy that is closest to him.

   (a) Prove that one of the boys does not get any ball.

   (b) Prove that none of the boys gets more than five balls.

5. Consider an infinite strip of unit squares. The squares are numbered 1, 2, 3, ... A pawn starts from one of the squares and moves according to the following rules:

   (i) The pawn cam move from the square n to the square $2n$ and vice-versa;

   (ii) The pawn can move from n to $3n + 1$ and vice-versa.

   Show that the pawn can reach the square 1 after finitely many steps.

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